If $x = 0,\ \sin^2\theta = 0 \Rightarrow \theta = 0$

$x = 1,\ \sin^2\theta = 1 \Rightarrow \theta = \pi/2$

$$\therefore \quad \beta(m, n) = \int_{\theta=0}^{\pi/2} (\sin^2\theta)^{m-1} (\cos^2\theta)^{n-1} \cdot 2\sin\theta\cos\theta\, d\theta$$

ie.,
$$\beta(m, n) = 2\int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta \qquad \ldots(3)$$

In (2) put $x = t^2$ $\therefore$ $dx = 2t\,dt$, t also varies from 0 to ∞

$$\therefore \quad \Gamma(n) = \int_{t=0}^{\infty} e^{-t^2} (t^2)^{n-1}\, 2t\, dt$$

ie.,
$$\Gamma(n) = 2\int_{0}^{\infty} e^{-t^2} t^{2n-1}\, dt \qquad \ldots(4)$$

(3) and (4) are also regarded as definitions of Beta function and Gamma function respectively.

5.32 Properties of Beta and Gamma functions

1. $\beta(m, n) = \beta(n, m)$

Proof : We have $\beta(m, n) = \int_{0}^{1} x^{m-1}(1-x)^{n-1}\, dx$

Put $x = 1 - y$ or $1 - x = y$ $\therefore$ $dx = -dy$

When $x = 0$, $y = 1$ and when $x = 1$, $y = 0$.

$$\therefore \quad \beta(m, n) = \int_{y=1}^{0} (1-y)^{m-1} y^{n-1} (-dy)$$

ie.,
$$\beta(m, n) = \int_{y=0}^{1} y^{n-1}(1-y)^{m-1}\, dy = \beta(n, m)$$

Thus $\beta(m, n) = \beta(n, m)$

2. *(i)* $\Gamma(n+1) = n\Gamma(n)$ *(ii)* $\Gamma(n+1) = n!$ *for a positive integer n*

Proof (i) By definition $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx$

$$\therefore \quad \Gamma(n+1) = \int_0^\infty e^{-x} x^n\, dx \text{ or } \int_0^\infty x^n e^{-x}\, dx$$

Integrating by parts we get,

$$\Gamma(n+1) = \left[x^n(-e^{-x}) \right]_0^\infty - \int_0^\infty (-e^{-x})\, n\, x^{n-1}\, dx$$

Note : $x^n/e^x \to 0$ *as* $x \to \infty$ *by L' Hospital's rule.*

$$\Gamma(n+1) = (0-0) + n\int_0^\infty e^{-x} x^{n-1}\, dx = n\Gamma(n)$$

$$\therefore \quad \boldsymbol{\Gamma(n+1) = n\Gamma(n)}$$

(ii) Continuing from the above, we have similarly

$$\Gamma(n) = (n-1)\Gamma(n-1),\ \Gamma(n-1) = (n-2)\Gamma(n-2)\ldots$$

$$\Gamma(3) = 2\Gamma(2),\ \Gamma(2) = 1\Gamma(1).$$

Now we have,

$$\Gamma(n+1) = n\{(n-1)\cdot\Gamma(n-1)\} = n(n-1)\{(n-2)\Gamma(n-2)\} \text{ etc.}$$

Thus $\Gamma(n+1) = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1\cdot \Gamma(1) = n!\,\Gamma(1)$

But $\Gamma(1) = \int_0^\infty e^{-x} x^0\, dx = -\left[e^{-x} \right]_0^\infty = -(0-1) = 1$

Thus $\boldsymbol{\Gamma(n+1) = n!}$ for a positive integer n.

Note *:*

1. *This result can be remembered in the form.* $\Gamma(n) = (n-1)\Gamma(n-1)$ *where* $n \neq 1$ *and* $\Gamma(n) = (n-1)!$ *where n is a positive integer.*
2. $\Gamma(n)$ *is not defined for* $n = 0$ *and also for a negative integer n.*
3. $\Gamma(n+1) = n\Gamma(n)$ *or* $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$ *and this expression is used for finding* $\Gamma(n)$ *when n is a negative real number.*

5.33 Relationship between Beta and Gamma functions

$$\beta(m, n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$$

Proof : We have by the definition of Beta and Gamma functions

$$\beta(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \qquad \ldots (1)$$

$$\Gamma(n) = 2 \int_0^{\infty} e^{-x^2} x^{2n-1} \, dx \qquad \ldots (2)$$

$$\Gamma(m) = 2 \int_0^{\infty} e^{-y^2} y^{2m-1} \, dy \qquad \ldots (3)$$

$$\Gamma(m+n) = 2 \int_0^{\infty} e^{-r^2} r^{2(m+n)-1} \, dr \qquad \ldots (4)$$

Now $\Gamma(m) \cdot \Gamma(n) = 4 \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} x^{2n-1} y^{2m-1} \, dx \, dy \qquad \ldots (5)$

Let us evaluate R.H.S by changing into polars.
Putting $x = r\cos\theta$, $y = r\sin\theta$, we have $x^2 + y^2 = r^2$
Also $dx\,dy = r\,dr\,d\theta$. r varies from 0 to ∞, θ varies from 0 to $\pi/2$
(*Analogous to Problem*-35).

We now have (5) in the form

$$\Gamma(m) \cdot \Gamma(n) = 4 \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} (r\cos\theta)^{2n-1} (r\sin\theta)^{2m-1} \, r \, dr \, d\theta$$

$$= 4 \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r^{2m+2n-1} \sin^{2m-1}\theta \cos^{2n-1}\theta \, dr \, d\theta$$

$$= \left[2 \int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1} \, dr \right] \left[2 \int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \right]$$

$\therefore$ $\Gamma(m) \cdot \Gamma(n) = \Gamma(m+n) \cdot \beta(m, n)$ by using (1) and (4).

Thus $\beta(m, n) = \dfrac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)}$

Corollary : *To show that* $\Gamma(1/2) = \sqrt{\pi}$

Putting $n = 1/2$ in this result we get,

$$\beta(1/2, 1/2) = \frac{\Gamma(1/2)\cdot\Gamma(1/2)}{\Gamma(1)} \quad \text{But } \Gamma(1) = 1$$

$$\therefore \quad \beta(1/2, 1/2) = \{\Gamma(1/2)\}^2 \qquad \ldots(6)$$

Now consider $\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$

$$\beta(1/2, 1/2) = 2\int_0^{\pi/2} \sin^0\theta\cos^0\theta\, d\theta = 2\int_0^{\pi/2} 1\, d\theta = 2\Big[\theta\Big]_0^{\pi/2} = \pi$$

Now we have from (6) $\pi = \{\Gamma(1/2)\}^2$

Thus $\boldsymbol{\Gamma(1/2) = \sqrt{\pi}}$

Note : *We can independently prove that* $\Gamma(1/2) = \sqrt{\pi}$. *The proof is as follows.*

We have by the definition $\Gamma(n) = 2\int_0^{\infty} e^{-x^2} x^{2n-1}\, dx$

$$\therefore \quad \Gamma(1/2) = 2\int_0^{\infty} e^{-x^2}\, dx = 2\int_0^{\infty} e^{-y^2}\, dy$$

Hence $\{\Gamma(1/2)\}^2 = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}\, dx\, dy = 4\cdot\frac{\pi}{4} = \pi$

(*We need to retrace the steps of Problem-35*)

Thus $\boldsymbol{\Gamma(1/2) = \sqrt{\pi}}$

5.34 Duplication formula

$$\sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\,\Gamma(m)\,\Gamma(m+1/2)$$

$$\beta(m, 1/2) = 2^{2m-1}\,\beta(m, m)$$

Proof : We have,

$$\frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)} = \beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta \qquad \ldots(1)$$

Putting $n = 1/2$ in (1) we get,

$$\frac{\Gamma(m)\Gamma(1/2)}{\Gamma(m+1/2)} = \beta(m,\ 1/2) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^0\theta \, d\theta$$

ie., $$\frac{\Gamma(m)\cdot\sqrt{\pi}}{\Gamma(m+1/2)} = \beta(m,\ 1/2) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta \qquad \ldots (2)$$

Also by putting $n = m$ in (1) we get,

$$\frac{\Gamma(m)\cdot\Gamma(m)}{\Gamma(2m)} = \beta(m,\ m) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2m-1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1} \, d\theta$$

$$= 2\int_0^{\pi/2} \left\{\frac{\sin 2\theta}{2}\right\}^{2m-1} d\theta$$

ie., $$\frac{\Gamma(m)\cdot\Gamma(m)}{\Gamma(2m)} = \beta(m,\ m) = 2\cdot\frac{1}{2^{2m-1}}\int_0^{\pi/2} \sin^{2m-1} 2\theta \, d\theta$$

In the integral, put $2\theta = \phi$ $\therefore$ $d\theta = d\phi/2$. ϕ varies from 0 to π

$\therefore$ $$\frac{\Gamma(m)\cdot\Gamma(m)}{\Gamma(2m)} = \beta(m,\ m) = 2\cdot\frac{1}{2^{2m-1}}\int_{\phi=0}^{\pi} \sin^{2m-1}\phi \cdot \frac{d\phi}{2}$$

ie., $$\frac{\Gamma(m)\cdot\Gamma(m)}{\Gamma(2m)} = \beta(m,\ m) = \frac{1}{2^{2m-1}}\int_0^{\pi} \sin^{2m-1}\phi \, d\phi$$

Using the fact that $\int_0^{\pi} \sin^k d\theta = 2\int_0^{\pi/2} \sin^k\theta \, d\theta$, we have,

$$\frac{\Gamma(m)\cdot\Gamma(m)}{\Gamma(2m)} = \beta(m,\ m) = \frac{1}{2^{2m-1}} 2\int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi$$

or

$$2^{2m-1}\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} = 2^{2m-1}\beta(m,\ m) = 2\int_0^{\pi/2} \sin^{2m-1}\phi \, d\phi \qquad \ldots (3)$$

With reference to (2) and (3) we can say that

$$2\int_0^{\pi/2}\sin^{2m-1}\theta\,d\theta = 2\int_0^{\pi/2}\sin^{2m-1}\phi\,d\phi$$ since the variable is arbitrary in a definite integral.

Thus we must have

$$\frac{\Gamma(m)\sqrt{\pi}}{\Gamma(m+1/2)} = 2^{2m-1}\frac{\Gamma(m)\Gamma(m)}{\Gamma(2m)} \text{ or } \beta(m,\,1/2) = 2^{2m-1}\beta(m,\,m)$$

$$\therefore \quad \sqrt{\pi}\,\Gamma(2m) = 2^{2m-1}\Gamma(m)\Gamma(m+1/2) \qquad \ldots(4)$$

or $$\beta(m,\,1/2) = 2^{2m-1}\beta(m,\,m) \qquad \ldots(5)$$

(4) is the duplication formula in terms of gamma functions and
(5) is the duplication formula in terms of beta functions.

Corollary : Putting $m = 1/4$ in (4) we get,

$$\sqrt{\pi}\,\Gamma(1/2) = 2^{-1/2}\Gamma(1/4)\Gamma(3/4)$$

ie., $$\sqrt{\pi}\cdot\sqrt{\pi} = 1/\sqrt{2}\cdot\Gamma(1/4)\Gamma(3/4)$$

Thus $$\Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}$$

WORKED PROBLEMS

Results to remember

(i) $\Gamma(n) = (n-1)\Gamma(n-1)$,
$\Gamma(n) = (n-1)!$ if n is a positive integer

(ii) $\beta(m,\,n) = \dfrac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$

(iii) $\Gamma(1) = 1,\ \Gamma(1/2) = \sqrt{\pi},\ \Gamma(1/4)\Gamma(3/4) = \pi\sqrt{2}$

---

39. *Show that* $\beta(m+1,\,n)+\beta(m,\,n+1) = \beta(m,\,n)$

>> Using the relationship between beta and gamma functions in L.H.S we get

$$\frac{\Gamma(m+1)\Gamma(n)}{\Gamma(m+n+1)} + \frac{\Gamma(m)\Gamma(n+1)}{\Gamma(m+n+1)} \quad \text{But } \Gamma(p+1) = p\,\Gamma(p)$$

$$= \frac{m\,\Gamma(m)\Gamma(n)}{(m+n)\Gamma(m+n)} + \frac{\Gamma(m)\,n\,\Gamma(n)}{(m+n)\Gamma(m+n)}$$

$$= \frac{\Gamma(m)\cdot\Gamma(n)}{(m+n)\Gamma(m+n)}(m+n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)} = \beta(m,\,n) = \text{R.H.S}$$

Note : *We can also obtain the result from the basic definition of Beta function.*

---

40. *Show that* $\beta(m, n)\beta(m+n, p) = \beta(n, p)\beta(n+p, m)$

>> Consider L.H.S = $\beta(m, n)\cdot\beta(m+n, p)$

ie., $$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\cdot\frac{\Gamma(m+n)\Gamma(p)}{\Gamma(m+n+p)} = \frac{\Gamma(m)\Gamma(n)\Gamma(p)}{\Gamma(m+n+p)} \quad \ldots(1)$$

Now consider R.H.S = $\beta(n, p)\beta(n+p, m)$

ie., $$= \frac{\Gamma(n)\Gamma(p)}{\Gamma(n+p)}\cdot\frac{\Gamma(n+p)\cdot\Gamma(m)}{\Gamma(m+n+p)} = \frac{\Gamma(m)\Gamma(n)\Gamma(p)}{\Gamma(m+n+p)} \quad \ldots(2)$$

Thus from (1) and (2) **L.H.S = R.H.S.**

---

41. *Show that* $\dfrac{\beta(m+2, n-2)}{m(m+1)} = \dfrac{\beta(m, n)}{(n-1)(n-2)}$

>> $$\text{L.H.S} = \frac{\beta(m+2, n-2)}{m(m+1)} = \frac{\Gamma(m+2)\Gamma(n-2)}{\Gamma(m+n)\cdot m(m+1)}$$

Using $\Gamma(n) = (n-1)\,\Gamma(n-1)$ we have,

$\Gamma(m+2) = (m+1)m\,\Gamma(m)$.

Also we shall multiply and divide $(n-1)\,(n-2)$

$$\text{L.H.S} = \frac{\Gamma(m)\cdot(n-1)(n-2)\Gamma(n-2)}{(n-1)(n-2)\Gamma(m+n)} = \frac{\Gamma(m)\Gamma(n)}{(n-1)(n-2)\Gamma(m+n)}$$

ie., $$\text{L.H.S} = \frac{\beta(m, n)}{(n-1)(n-2)} = \textbf{R.H.S}$$

Thus **L.H.S. = R.H.S**

---

42. *Evaluate* (i) $\dfrac{\Gamma(3)\Gamma(2\cdot5)}{\Gamma(5\cdot5)}$ (ii) $\dfrac{6\Gamma(8/3)}{5\Gamma(2/3)}$

(iii) $\Gamma(-7/2)$ (iv) $\beta(7/2, -1/2)$

>> (i) $$\frac{\Gamma(3)\Gamma(2\cdot5)}{\Gamma(5\cdot5)} = \frac{(2!)(1\cdot5)(0\cdot5)\Gamma(0\cdot5)}{(4\cdot5)(3\cdot5)(2\cdot5)(1\cdot5)(0\cdot5)\Gamma(0\cdot5)}$$

$$= \frac{2}{(4\cdot5)(3\cdot5)(2\cdot5)} = \frac{2}{9/2\cdot7/2\cdot5/2} = \mathbf{\frac{16}{315}}$$

(ii) $$\frac{6\Gamma(8/3)}{5\Gamma(2/3)} = \frac{6\cdot5/3\cdot2/3\cdot\Gamma(2/3)}{5\Gamma(2/3)} = \mathbf{\frac{4}{3}}$$

(iii) To find $\Gamma(-7/2)$ we consider the relation in the form

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\therefore \quad \Gamma(-7/2) = \frac{\Gamma(-5/2)}{-7/2} = \frac{-2}{7} \cdot \frac{\Gamma(-3/2)}{-5/2} = \frac{4}{35} \frac{\Gamma(-1/2)}{-3/2}$$

ie., $$\Gamma(-7/2) = \frac{-8}{105} \frac{\Gamma(1/2)}{-1/2} = \frac{16}{105} \sqrt{\pi}$$

(iv) $$\beta(7/2, -1/2) = \frac{\Gamma(7/2) \cdot \Gamma(-1/2)}{\Gamma(3)}$$

$$= \frac{\{5/2 \cdot 3/2 \cdot 1/2\, \Gamma(1/2)\} \cdot \frac{\Gamma(1/2)}{-1/2}}{2!}$$

$$\beta(7/2, -1/2) = \frac{(15/8)\sqrt{\pi} \cdot -2\sqrt{\pi}}{2} = \frac{-15\pi}{8}$$

---

43. *If n is a positive integer or a negative non integer and k is any positive integer, prove that*

$$\Gamma(n+k+1) = n(n+1)(n+2)\cdots(n+k)\Gamma(n)$$

>> **Note** : *We have to first establish the result* $\Gamma(n+1) = n\Gamma(n)$ ... (1)

We shall establish the desired result by the principle of mathematical induction.

Step-1. We shall verify the result for $k = 1$

When $k = 1$, $\Gamma(n+k+1) = \Gamma(n+2) = (n+1)n\,\Gamma(n)$ by (1)

ie., $\Gamma(\overline{n+1}+1) = n(n+1)\Gamma(n)$

This shows that the result is true for $n = 1$

Step-2. We shall assume the result to be true for $k = m$

$$\Gamma(n+m+1) = n(n+1)(n+2)\cdots(n+m)\Gamma(n) \quad \ldots (2)$$

Step-3. We shall prove the result for $k = m+1$

$\therefore$ $\Gamma(n+\overline{m+1}+1) = \Gamma[(n+m+1)+1]$

ie., $= (n+m+1)\Gamma(n+m+1)$ by (1)

$= (n+m+1)\{n(n+1)(n+2)\cdots(n+m)\Gamma(n)\}$ by using (2)

ie., $$\Gamma(n+\overline{m+1}+1) = n(n+1)(n+2)\cdots(n+m)(n+\overline{m+1})\Gamma(n) \quad \ldots (3)$$

Comparing (2) and (3) we can say that the result is true for $k = m+1$.

Hence by the principle of mathematical induction the result is true for all positive integral values of k.

Thus $\boldsymbol{\Gamma(n+k+1) = n(n+1)(n+2)\cdots(n+k)\Gamma(n)}$

---

44. *If n is a positive integer, prove that* $\Gamma(n+1/2) = \dfrac{1\cdot3\cdot5\cdots 2n-1}{2^n}\sqrt{\pi}$ *and hence find* $\Gamma(9/2)$

>> We shall prove the result by mathematical induction .

When $n = 1$, L.H.S $= \Gamma(1+1/2) = 1/2\,\Gamma(1/2) = 1/2\cdot\sqrt{\pi}$ = R.H.S

$\therefore$ the result is true for $n = 1$

We shall assume the result to be true for $n = m$

ie., $$\Gamma(m+1/2) = \frac{1\cdot3\cdot5\cdots 2m-1}{2^m}\sqrt{\pi} \qquad \ldots(1)$$

Now $$\Gamma(\overline{m+1}+1/2) = \Gamma(\overline{m+1/2}+1)$$
$$= (m+1/2)\,\Gamma(m+1/2) \quad \because \Gamma(n+1) = n\,\Gamma(n)$$

$\therefore$ $$\Gamma(\overline{m+1}+1/2) = (m+1/2)\cdot\frac{1\cdot3\cdot5\cdots 2m-1}{2^m}\sqrt{\pi}, \text{ by using (1)}$$

$$= \frac{2m+1}{2}\cdot\frac{1\cdot3\cdot5\cdots 2m-1}{2^m}\sqrt{\pi}$$

$$= \frac{[2(m+1)-1]}{2}\cdot\frac{1\cdot3\cdot5\cdots 2m-1}{2^m}\sqrt{\pi}$$

ie., $$\Gamma(\overline{m+1}+1/2) = \frac{1\cdot3\cdot5\cdots(2m-1)[2(m+1)-1]}{2^{m+1}}\sqrt{\pi} \qquad \ldots(2)$$

Comparing (1) and (2) we can say that the result is true for $n = m+1$

Hence by the principle of induction, the result is true for all positive integral values of n.

Thus $$\boldsymbol{\Gamma(n+1/2) = \frac{1\cdot3\cdot5\cdots 2n-1}{2^n}\sqrt{\pi}}$$

Next we shall find $\Gamma(9/2)$

$\Gamma(9/2) = \Gamma(4+1/2)$ and by putting $n = 4$ in the result we get

$$\Gamma(9/2) = \frac{1\cdot3\cdot5\cdot7}{2^4}\sqrt{\pi} = \frac{\mathbf{105\sqrt{\pi}}}{\mathbf{16}}$$

---

45. *Prove that* $\int_0^\infty e^{-ax}x^{n-1}\,dx = \dfrac{\Gamma(n)}{a^n}$ *where* a *and* n *are positive and hence deduce that*

(i) $\int_0^\infty e^{-ax}x^{n-1}\cos bx\,dx = \dfrac{\Gamma(n)}{r^n}\cos n\theta$

(ii) $\int_0^\infty e^{-ax}x^{n-1}\sin bx\,dx = \dfrac{\Gamma(n)}{r^n}\sin n\theta$

where $r^2 = a^2+b^2$ *and* $\theta = \tan^{-1}(b/a)$

\>> Consider $I = \int_0^\infty e^{-ax}x^{n-1}\,dx$

Put $ax = t$ $\therefore$ $dx = dt/a$ and t also varies from 0 to ∞

$$\text{Hence } I = \int_{t=0}^\infty e^{-t}\left(\frac{t}{a}\right)^{n-1}\frac{dt}{a} = \frac{1}{a^n}\int_0^\infty e^{-t}t^{n-1}\,dt = \frac{\Gamma(n)}{a^n}$$

$$\text{Thus } \int_0^\infty e^{-ax}x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}$$

Now let us replace a by $a+ib$

$$\therefore \quad \int_0^\infty e^{-(a+ib)x}x^{n-1}\,dx = \frac{\Gamma(n)}{(a+ib)^n}$$

$$ie., \quad \int_0^\infty e^{-ax}(\cos bx - i\sin bx)\,x^{n-1}\,dx = \frac{\Gamma(n)}{(a+ib)^n} \qquad \ldots (1)$$

We shall express R.H.S in the form $p+iq$ by using the substitution

$a = r\cos\theta$, $b = r\sin\theta$.

This gives us $a^2+b^2 = r^2$ and $b/a = \tan\theta$ or $\theta = \tan^{-1}(b/a)$

$$\text{Hence } \frac{\Gamma(n)}{(a+ib)^n} = \frac{\Gamma(n)}{r^n e^{in\theta}} = \frac{\Gamma(n)}{r^n}e^{-in\theta}$$

$$ie., \quad \frac{\Gamma(n)}{(a+ib)^n} = \frac{\Gamma(n)}{r^n}(\cos n\theta - i\sin n\theta) \qquad \ldots (2)$$

Using (2) in the R.H.S of (1) we get,

$$\int_0^\infty e^{-ax}(\cos bx - i\sin bx)\, x^{n-1}\, dx = \frac{\Gamma(n)}{r^n}(\cos n\theta - i\sin n\theta)$$

Thus by equating the real and imaginary parts we get,

$$\int_0^\infty e^{-ax} x^{n-1} \cos bx\, dx = \frac{\Gamma(n)}{r^n}\cos n\theta \quad \text{and} \quad \int_0^\infty e^{-ax} x^{n-1} \sin bx\, dx = \frac{\Gamma(n)}{r^n}\sin n\theta$$

where $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}(b/a)$

---

46. *Prove that :* (i) $\displaystyle\int_0^\infty x^{n-1}\cos ax\, dx = \frac{\Gamma(n)}{a^n}\cos(n\pi/2)$

(ii) $\displaystyle\int_0^\infty x^{n-1}\sin ax\, dx = \frac{\Gamma(n)}{a^n}\sin(n\pi/2)$

\>> We know that $e^{-iax} = \cos ax - i\sin ax$

Let us consider $I = \displaystyle\int_0^\infty x^{n-1} e^{-iax}\, dx$

Put $iax = t \quad \therefore \quad dx = dt/ia, \quad t$ varies from 0 to ∞

Hence $I = \displaystyle\int_0^\infty \left(\frac{t}{ia}\right)^{n-1} e^{-t}\, \frac{dt}{ia} = \frac{1}{i^n a^n}\int_0^\infty e^{-t}\, t^{n-1}\, dt$

$$\therefore \quad \int_0^\infty x^{n-1} e^{-iax}\, dx = \frac{1}{i^n a^n}\Gamma(n) \qquad \ldots (1)$$

In order to express R.H.S in the form $p + iq$,

let $\quad i = r(\cos\theta + i\sin\theta)$

$\therefore \quad r\cos\theta = 0,\ r\sin\theta = 1 \Rightarrow r^2 = 1,\ \tan\theta = \infty$ or $\theta = \pi/2$

Hence $i = \cos(\pi/2) + i\sin(\pi/2) = e^{i\pi/2}$

$$\text{Thus} \quad \frac{1}{i^n a^n}\Gamma(n) = \frac{\Gamma(n)}{e^{in\pi/2}\cdot a^n} = \frac{\Gamma(n)}{a^n}e^{-in\pi/2} \qquad \ldots (2)$$

Using (2) in the R.H.S of (1) we have,

$$\int_0^\infty x^{n-1} e^{-iax}\, dx = \frac{\Gamma(n)}{a^n} e^{-in\pi/2}$$

ie., $$\int_0^\infty x^{n-1} (\cos ax - i \sin ax)\, dx = \frac{\Gamma(n)}{a^n} \{ \cos(n\pi/2) - i \sin(n\pi/2) \}$$

Thus by equating the real and imaginary parts we get,

$$\int_0^\infty x^{n-1} \cos ax\, dx = \frac{\Gamma(n)}{a^n} \cos(n\pi/2) \quad \text{and} \quad \int_0^\infty x^{n-1} \sin ax\, dx = \frac{\Gamma(n)}{a^n} \sin(n\pi/2)$$

---

47. *Show that* $\int_0^1 \left[\log(1/y) \right]^{n-1} dy = \Gamma(n)$

\>> Consider L.H.S and put $\log(1/y) = t$

ie., $1/y = e^t$ or $y = e^{-t}$ $\therefore$ $dy = -e^{-t}\, dt$

If $y = 0,\ e^{-t} = 0 \Rightarrow t = \infty$; If $y = 1,\ e^{-t} = 1 \Rightarrow t = 0$

$$\therefore \quad \text{L.H.S} = \int_{t=\infty}^{0} t^{n-1} (-e^{-t})\, dt = \int_0^\infty e^{-t} t^{n-1}\, dt = \Gamma(n) = \text{R.H.S}$$

---

48. *Show that* $\int_0^1 x^m [\log(1/x)]^n dx = \frac{n!}{(m+1)^{n+1}}$ *where n is a positive integer.*

\>> $$\text{L.H.S} = \int_0^1 x^m [\log(1/x)]^n dx$$

Put $\log(1/x) = t$ or $1/x = e^t$ *ie.,* $x = e^{-t}$ $\therefore$ $dx = -e^{-t}\, dt$

If $x = 0,\ e^{-t} = 0 \Rightarrow t = \infty$; If $x = 1,\ e^{-t} = 1 \Rightarrow t = 0$

$$\text{L.H.S} = \int_{t=\infty}^{0} (e^{-t})^m\, t^n (-e^{-t})\, dt$$

$$\text{L.H.S} = \int_0^\infty (e^{-t})^{m+1} t^n \, dt = \int_0^\infty e^{-(m+1)t} t^n \, dt \qquad \ldots (1)$$

Now, let $(m+1)t = u \quad \therefore \quad dt = du/(m+1)$

u also varies from 0 to ∞ and hence (1) becomes

$$\text{L.H.S} = \int_{u=0}^\infty e^{-u} \left(\frac{u}{m+1} \right)^n \frac{du}{(m+1)} = \frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-u} u^n \, du$$

$$\text{L.H.S} = \frac{1}{(m+1)^{n+1}} \cdot \Gamma(n+1) = \frac{n!}{(m+1)^{n+1}} = \text{R.H.S}$$

Evaluation of definite integrals by converting into gamma functions

- We need to take a suitable substitution keeping in mind the definition of gamma function in two of the standard forms :

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx \quad ; \quad \Gamma(n) = 2 \int_0^\infty e^{-x^2} x^{2n-1} \, dx$$

- We need to correlate the value corresponding to $n-1$ or $2n-1$ as the case may be and find n.

49. *Show that* $\displaystyle\int_0^\infty \sqrt{y}\, e^{-y^2} dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy = \frac{\pi}{2\sqrt{2}}$

Let
$$I_1 = \int_0^\infty \sqrt{y}\, e^{-y^2} dy = \int_0^\infty e^{-y^2} y^{1/2} dy \qquad \ldots (1)$$

$$I_2 = \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dy = \int_0^\infty e^{-y^2} y^{-1/2} dy \qquad \ldots (2)$$

We have
$$\frac{1}{2}\Gamma(n) = \int_0^\infty e^{-x^2} x^{2n-1} \, dx \qquad \ldots (3)$$

Comparing (1) and (3) we have $2n-1 = 1/2 \Rightarrow n = 3/4$

$\therefore \quad I_1 = 1/2 \cdot \Gamma(3/4)$

Comparing (2) and (3) we have $2n-1 = -1/2 \Rightarrow n = 1/4$

$\therefore \quad I_2 = 1/2 \cdot \Gamma(1/4)$

Hence the required $I_1 \times I_2 = \frac{1}{4}\Gamma(3/4)\Gamma(1/4)$

But $\quad \Gamma(3/4) \cdot \Gamma(1/4) = \pi\sqrt{2}$

Thus $\quad I_1 \times I_2 = \frac{\pi\sqrt{2}}{4} = \frac{\pi}{2\sqrt{2}}$

---

50. *Show that* $\int_0^\infty x\,e^{-x^8}\,dx \times \int_0^\infty x^2\,e^{-x^4}\,dx = \frac{\pi}{16\sqrt{2}}$

>> Let $\quad I_1 = \int_0^\infty x\,e^{-x^8}\,dx$

Put $\quad x^8 = t \;\therefore\; 8x^7\,dx = dt$ or $x\,dx = dt/8x^6$

ie., $\quad x\,dx = dt/8(t^{1/8})^6$ or $x\,dx = dt/8t^{3/4}$. t also varies from 0 to ∞

$$\therefore \quad I_1 = \int_{t=0}^\infty e^{-t}\frac{dt}{8t^{3/4}} = \frac{1}{8}\int_0^\infty e^{-t}\,t^{-3/4}\,dt$$

We have $\quad \Gamma(n) = \int_0^\infty e^{-x}\,x^{n-1}\,dx \qquad \ldots(1)$

Taking $(n-1) = -3/4$ we get $n = 1/4$

Hence $\quad I_1 = 1/8 \cdot \Gamma(1/4) \qquad \ldots(2)$

Let $\quad I_2 = \int_0^\infty x^2\,e^{-x^4}\,dx$

Put $x^4 = u \;\therefore\; 4x^3\,dx = du,\; x^2\,dx = du/4x$ or $x^2\,dx = du/4u^{1/4}$

u also varies from 0 to ∞

$$\therefore \quad I_2 = \int_0^\infty e^{-u}\frac{du}{4u^{1/4}} = \frac{1}{4}\int_0^\infty e^{-u}\,u^{-1/4}\,du$$

Comparing with (1) we must have $(n-1) = -1/4$ or $n = 3/4$

Hence $\quad I_2 = 1/4 \cdot \Gamma(3/4) \qquad \ldots(3)$

Now from (2) and (3) $I_1 \times I_2 = \frac{1}{32}\Gamma(1/4)\Gamma(3/4) = \frac{\pi\sqrt{2}}{32} = \frac{\pi}{16\sqrt{2}}$

Thus $\quad I_1 \times I_2 = \frac{\pi}{16\sqrt{2}}$

---

51. *Show that* $\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\, dt = \sqrt{\pi/s}\,,\ s > 0$

>> Let $I = \int_0^\infty \frac{e^{-st}}{\sqrt{t}}\, dt$

Put $st = x \;\therefore\; dt = dx/s$ and x varies from 0 to ∞

$$\therefore \quad I = \int_{x=0}^{\infty} \frac{e^{-x}}{\sqrt{x/s}} \cdot \frac{dx}{s} = \frac{1}{\sqrt{s}} \int_0^\infty e^{-x}\, x^{-1/2}\, dx$$

$$I = \frac{1}{\sqrt{s}} \int_0^\infty e^{-x}\, x^{1/2-1}\, dx = \frac{1}{\sqrt{s}}\Gamma(1/2) = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\pi/s}$$

Thus $I = \sqrt{\pi/s}$

---

52. *Show that* $\Gamma(n) = 2a^n \int_0^\infty x^{2n-1}\, e^{-ax^2}\, dx$

>> Consider R.H.S and put $ax^2 = t \;\therefore\; 2ax\, dx = dt$

$$dx = \frac{dt}{2ax} = \frac{dt}{2a\sqrt{t/a}} = \frac{dt}{2\sqrt{at}} \quad \text{and } t \text{ varies from 0 to } \infty$$

$$\text{R.H.S} = 2a^n \int_{t=0}^{\infty} [\sqrt{t/a}\,]^{2n-1}\, e^{-t}\, \frac{dt}{2\sqrt{at}}$$

$$= a^n \int_0^\infty [\sqrt{t/a}\,]^{2n}\, [\sqrt{t/a}\,]^{-1}\, e^{-t}\, \frac{dt}{\sqrt{at}}$$

$$\text{R.H.S} = a^n \int_0^\infty \left(\frac{t}{a}\right)^n \sqrt{\frac{a}{t}}\; e^{-t}\, \frac{dt}{\sqrt{at}} = \int_0^\infty e^{-t}\, t^{n-1}\, dt = \Gamma(n) = \text{L.H.S}$$

---

53. *Show that* $\int_a^{\infty} e^{(2ax-x^2)}\,dx = \frac{\sqrt{\pi}}{2}\,e^{a^2}$

>> Let $I = \int_a^{\infty} e^{(2ax-x^2)}\,dx = \int_a^{\infty} e^{-(x^2-2ax)}\,dx$

$$= \int_a^{\infty} e^{-[(x-a)^2-a^2]}\,dx = e^{a^2}\int_a^{\infty} e^{-(x-a)^2}\,dx$$

Put $x-a = t$ $\therefore$ $dx = dt$. If $x = a,\ t = 0$ and if $x = \infty,\ t = \infty$.

Hence $I = e^{a^2}\int_{t=0}^{\infty} e^{-t^2}\,dt$ . . . (1)

But $\frac{1}{2}\Gamma(n) = \int_0^{\infty} e^{-x^2} x^{2n-1}\,dx$

Putting $n = 1/2$ we get $1/2\cdot\Gamma(1/2) = \int_0^{\infty} e^{-x^2}\,dx$

$\therefore$ $\sqrt{\pi}/2 = \int_0^{\infty} e^{-x^2}\,dx = \int_0^{\infty} e^{-t^2}\,dt$ . . . (2)

Thus by using (2) in (1) we get $\boldsymbol{I = \sqrt{\pi}\,e^{a^2}/2}$

Evaluate the following integrals

54. $\int_0^{\infty} x^{3/2} e^{-x}\,dx$

55. $\int_0^{\infty} x^{1/4} e^{-\sqrt{x}}\,dx$

56. $\int_0^{\infty} x^6 e^{-2x}\,dx$

57. $\int_0^1 [\log(1/x)]^{3/2}\,dx$

58. $\int_0^1 x^2 [\log(1/x)]^3\,dx$

59. $\int_0^1 \sqrt{-\log x}\,dx$

60. $\int_0^{\infty} 2^{-3x^2}\,dx$

61. $\int_0^{\infty} x^{-3/2}(1-e^{-x})\,dx$

62. $\int_0^1 (\log x)^3\,dx$

63. $\int_0^1 (x\log x)^4\,dx$

54. Let $I = \int_0^{\infty} x^{3/2} e^{-x} dx$

We have $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$

Taking $(n-1) = 3/2$ we get $n = 5/2$

Thus $I = \Gamma(5/2) = 3/2 \cdot 1/2 \cdot \Gamma(1/2) = \mathbf{3\sqrt{\pi}/4}$

---

55. Let $I = \int_0^{\infty} x^{1/4} e^{-\sqrt{x}} dx.$

Put $\sqrt{x} = t$ or $x = t^2$ $\therefore$ $dx = 2t\,dt$ and t varies from 0 to ∞

Hence $I = \int_{t=0}^{\infty} (t^2)^{1/4} e^{-t} 2t\,dt = 2 \int_0^{\infty} e^{-t} t^{3/2} dt = 2 \int_0^{\infty} e^{-t} t^{(5/2)-1} dt$

Thus $I = 2\Gamma(5/2) = 2 \cdot 3/2 \cdot 1/2\, \Gamma(1/2) = \mathbf{3\sqrt{\pi}/2}$

---

56. Let $I = \int_0^{\infty} x^6 e^{-2x} dx$

Put $2x = t$ $\therefore$ $dx = dt/2$ and t varies from 0 to ∞

$\therefore \quad I = \int_{t=0}^{\infty} \left(\frac{t}{2}\right)^6 e^{-t} \frac{dt}{2} = \frac{1}{2^7} \int_0^{\infty} e^{-t} t^6 dt = \frac{1}{128} \int_0^{\infty} e^{-t} t^{7-1} dt$

Thus $I = \frac{\Gamma(7)}{128} = \frac{6!}{128} = \frac{720}{128} = \mathbf{\frac{45}{8}}$

---

57. Let $I = \int_0^{1} [\log(1/x)]^{3/2} dx$

Note : *This problem is a particular case of problem-52 and can be proceeded on the same lines to obtain the result independently.* $(n-1) = 3/2$ $\therefore$ $n = 5/2$

$I = \Gamma(5/2) = 3/2 \cdot 1/2\, \Gamma(1/2) = \mathbf{3\sqrt{\pi}/4}$

---

58. Let $I = \int_0^1 x^2 [\log(1/x)]^3\, dx$

Note : *This example is a particular case of problem-53. and can be proceeded on the same lines to obtain the result independently. Here* $m = 2$, $n = 3$

$\therefore \quad n!/(m+1)^{n+1} = 3!/3^4 = 2/27$

Thus $\quad \mathbf{I = 2/27}$

---

59. Let $I = \int_0^1 \sqrt{-\log x}\, dx$

Put $-\log x = t$ or $\log x = -t$ *ie.,* $x = e^{-t}$ $\therefore$ $dx = -e^{-t}\, dt$

If $x = 0$, $e^{-t} = 0 \Rightarrow t = \infty$ and if $x = 1$, $e^{-t} = 1$ or $t = 0$

$$\therefore \quad I = \int_{t=\infty}^{0} \sqrt{t}\,(-e^{-t})\, dt = \int_0^{\infty} e^{-t} t^{1/2}\, dt = \int_0^{\infty} e^{-t} t^{(3/2)-1}\, dt$$

Thus $\quad I = \Gamma(3/2) = 1/2 \cdot \Gamma(1/2) = \mathbf{\sqrt{\pi}/2}$

---

60. Let $I = \int_0^{\infty} 2^{-3x^2}\, dx = \int_0^{\infty} (e^{\log 2})^{-3x^2}\, dx = \int_0^{\infty} (e^{-3\log 2})^{x^2}\, dx$

Taking $k = -3\log 2$, $I = \int_0^{\infty} e^{-kx^2}\, dx$ ...(1)

Put $kx^2 = t$ $\therefore$ $2kx\, dx = dt$ or $dx = dt/2kx$

ie., $\quad dx = \dfrac{dt}{2k\sqrt{t/k}} = \dfrac{dt}{2\sqrt{k}\sqrt{t}}$ and t varies from 0 to ∞

$$\therefore \text{ (1) becomes } I = \int_{t=0}^{\infty} e^{-t} \frac{dt}{2\sqrt{k}\sqrt{t}} = \frac{1}{2\sqrt{k}} \int_0^{\infty} e^{-t} t^{-1/2}\, dt$$

$$I = \frac{1}{2\sqrt{k}} \int_0^{\infty} e^{-t} t^{(1/2)-1}\, dt = \frac{\Gamma(1/2)}{2\sqrt{k}} = \frac{\sqrt{\pi}}{2\sqrt{k}}$$

Thus $$I = \frac{\sqrt{\pi}}{2\sqrt{3\log 2}} = \frac{\sqrt{\pi}}{2\sqrt{\log 8}} = \frac{1}{2}\sqrt{\pi/\log 8}$$

---

61. Let $I = \int_0^\infty x^{-3/2}(1-e^{-x})\,dx = \int_0^\infty (1-e^{-x})\,x^{-3/2}\,dx$

Integrating by parts we have,

$$I = \left[(1-e^{-x})\,x^{-1/2}/(-1/2)\right]_0^\infty - \int_0^\infty \frac{x^{-1/2}}{-1/2}\cdot e^{-x}\,dx$$

$$= (0-0)+2\int_0^\infty e^{-x}x^{-1/2}\,dx = 2\,\Gamma(1/2)$$

Thus **$I = 2\sqrt{\pi}$**

---

62. Let $I = \int_0^1 (\log x)^3\,dx$

Put $\log x = -t$ or $x = e^{-t}$ $\therefore$ $dx = -e^{-t}\,dt$ and t varies from ∞ to 0

$$\therefore \quad I = \int_\infty^0 (-t)^3(-e^{-t})\,dt = -\int_0^\infty e^{-t}t^3\,dt = -\Gamma(4) = -3! = -6$$

Thus **$I = -6$**

---

63. Let $I = \int_0^1 (x\log x)^4\,dx = \int_0^1 x^4(\log x)^4\,dx$

Put $\log x = -t$ or $x = e^{-t}$ $\therefore$ $dx = -e^{-t}\,dt$, t varies from ∞ to 0

$$\therefore \quad I = \int_\infty^0 (e^{-t})^4(-t)^4(-e^{-t}\,dt) = \int_0^\infty e^{-5t}t^4\,dt$$

Put $5t = u$ $\therefore$ $dt = du/5$ and u varies from 0 to ∞

Hence $$I = \int_{u=0}^\infty e^{-u}\left(\frac{u}{5}\right)^4\cdot\frac{du}{5} = \frac{1}{5^5}\int_0^\infty e^{-u}u^4\,du = \frac{\Gamma(5)}{5^5} = \frac{4!}{5^5}$$

Thus **$I = 24/3125$**

Problems on converting an integral into beta function and evaluation by transforming into gamma function

We have $$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

Putting $2m - 1 = p$ and $2n - 1 = q$ we get

$$m = (p+1)/2, \; n = (q+1)/2$$

$$\therefore \quad \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$\therefore \quad \int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \qquad \ldots (1)$$

Integrals involving trigonometric functions are converted into the above form and in turn is expressed in terms of beta function. Further some of the integrals involving algebriac functions are converted into the trigonometric form by a suitable substitution. Two important forms along with the substitution is as follows.

(i) $(a - x^n)$: Substitution : $x^n = a\sin^2\theta$

(ii) $(a + x^n)$: Substitution : $x^n = a\tan^2\theta$

The integral expressed in terms of beta function is converted into gamma function by using the relation $\beta(m, n) = \Gamma(m)\cdot\Gamma(n)/\Gamma(m+n)$ for the purpose of evaluation.

Remark : *We have discussed the evaluation of the integral in (1) with the help of reduction formulae which is applicable only when p and q are positive integers. But the relation (1) is applicable in general when p and q are real numbers.*

64. *Show that* $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$

>> Let $$I_1 = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \int_0^{\pi/2} \sin^{-1/2}\theta \, d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta \, d\theta$$

and $$I_2 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta \, d\theta.$$

We have $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

Hence $I_1 = \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$

$I_2 = \frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$

$\therefore \quad I_1 \times I_2 = \frac{1}{4}\beta\left(1/4, 1/2\right)\cdot\beta\left(3/4, 1/2\right)$

ie., $= \frac{1}{4}\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)}\cdot\frac{\Gamma(3/4)\cdot\Gamma(1/2)}{\Gamma(5/4)}$

$= \frac{1}{4}\Gamma(1/4)\sqrt{\pi}\cdot\frac{\sqrt{\pi}}{1/4\cdot\Gamma(1/4)} = \pi$

Thus $\boldsymbol{I_1 \times I_2 = \pi}$

---

65. *Evaluate* $\int_0^{\pi/2}\sqrt{\cot\theta}\, d\theta$ *by expressing in terms of gamma functions.*

>> Let $I = \int_0^{\pi/2}\sqrt{\cot\theta}\, d\theta = \int_0^{\pi/2}\frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}}\, d\theta = \int_0^{\pi/2}\sin^{-1/2}\theta\cos^{1/2}\theta\, d\theta$

Hence $I = \frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{1/2+1}{2}\right)$

ie., $I = \frac{1}{2}\beta\left(1/4, 3/4\right) = \frac{1}{2}\frac{\Gamma(1/4)\cdot\Gamma(3/4)}{\Gamma(1)} = \frac{1}{2}\cdot\frac{\pi\sqrt{2}}{1} = \frac{\pi}{\sqrt{2}}$

Thus $\boldsymbol{I = \pi/\sqrt{2}}$

Note : $\int_0^{\pi/2}\sqrt{\tan\theta}\, d\theta$ *is also equal to* $\frac{\pi}{\sqrt{2}}$

---

66. *Show that* $\int_0^{\pi/2} \sin^p \theta \, d\theta \times \int_0^{\pi/2} \sin^{p+1} \theta \, d\theta = \dfrac{\pi}{2(p+1)}$

>> Let $I_1 = \int_0^{\pi/2} \sin^p \theta \, d\theta = \int_0^{\pi/2} \sin^p \theta \cos^0 \theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{1}{2}\right)$

$$I_2 = \int_0^{\pi/2} \sin^{p+1} \theta \, d\theta = \int_0^{\pi/2} \sin^{p+1} \theta \cos^0 \theta \, d\theta = \frac{1}{2} \beta\left(\frac{p+2}{2}, \frac{1}{2}\right)$$

$$\therefore \quad I_1 \times I_2 = \frac{1}{4} \cdot \frac{\Gamma(\overline{p+1}/2) \cdot \Gamma(1/2)}{\Gamma(\overline{p+2}/2)} \cdot \frac{\Gamma(\overline{p+2}/2) \cdot \Gamma(1/2)}{\Gamma(\overline{p+3}/2)}$$

ie.,
$$= \frac{\pi}{4} \frac{\Gamma(\overline{p+1}/2)}{\Gamma[\overline{p+1}/2+1]} = \frac{\pi}{4} \frac{\Gamma(\overline{p+1}/2)}{(\overline{p+1}/2)\Gamma(\overline{p+1}/2)} = \frac{\pi}{2(p+1)}$$

Thus $\boldsymbol{I_1 \times I_2 = \pi / 2(p+1)}$

Evaluate each of the following integrals by expressing in terms of gamma functions and verify the answer by applying the reduction formulae.

67. $\int_0^{\pi/2} \sin^6 \theta \, d\theta$ 68. $\int_0^{\pi/2} \cos^7 \theta \, d\theta$ 69. $\int_0^{\pi/2} \sin^4 \theta \cos^3 \theta \, d\theta$

67. Let $I = \int_0^{\pi/2} \sin^6 \theta \, d\theta = \int_0^{\pi/2} \sin^6 \theta \cos^0 \theta \, d\theta$

Hence $I = \frac{1}{2} \beta\left(\frac{7}{2}, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma(7/2) \cdot \Gamma(1/2)}{\Gamma(4)}$

$$I = \frac{1}{2} \cdot \frac{5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma(1/2)}{3!} \cdot \Gamma(1/2) = \frac{5\pi}{32}$$

Thus $\boldsymbol{I = 5\pi / 32}$

By reduction formula : $\int_0^{\pi/2} \sin^6 \theta \, d\theta = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \boldsymbol{\frac{5\pi}{32}}$

---

68. Let $I = \int_0^{\pi/2} \cos^7 \theta \, d\theta = \int_0^{\pi/2} \sin^0 \theta \cos^7 \theta \, d\theta$

Hence $I = \frac{1}{2} \beta\left(\frac{1}{2}, 4\right) = \frac{1}{2} \frac{\Gamma(1/2)\Gamma(4)}{\Gamma(9/2)}$

$$I = \frac{1}{2} \cdot \frac{\sqrt{\pi} \cdot 3!}{7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \cdot \sqrt{\pi}} = \frac{6}{2 \cdot (105/16)} = \frac{16}{35}$$

Thus $\mathbf{I = 16/35}$

By reduction formula : $\int_0^{\pi/2} \cos^7 \theta \, d\theta = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \mathbf{\frac{16}{35}}$

---

69. Let $I = \int_0^{\pi/2} \sin^4 \theta \cos^3 \theta \, d\theta = \frac{1}{2} \beta\left(\frac{5}{2}, 2\right)$

$$I = \frac{1}{2} \frac{\Gamma(5/2) \cdot \Gamma(2)}{\Gamma(9/2)} = \frac{1}{2} \cdot \frac{\Gamma(5/2) \cdot 1!}{7/2 \cdot 5/2\, \Gamma(5/2)} = \frac{2}{35}$$

Thus $\mathbf{I = 2/35}$

By reduction formula : $\int_0^{\pi/2} \sin^4 \theta \cos^3 \theta \, d\theta = \frac{(3 \cdot 1)(2)}{7 \cdot 5 \cdot 3 \cdot 1} = \mathbf{\frac{2}{35}}$

Express the following integrals in terms of beta function and hence evaluate them.

70. $\int_0^1 x^{3/2} (1-x)^{1/2} \, dx$

71. $\int_0^1 \sqrt{(1-x)/x} \, dx$

72. $\int_0^2 (4-x^2)^{3/2}$

73. $\int_0^4 x^{3/2} (4-x)^{5/2} \, dx$

74. $\int_0^a x^4 \sqrt{a^2 - x^2} \, dx$

75. $\int_0^2 \frac{x^2}{\sqrt{2-x}} \, dx$

76. $\int_{-1}^{1} \sqrt{1+x/1-x}\, dx$

77. $\int_{0}^{\infty} \frac{dx}{1+x^4}$

78. $\int_{0}^{\infty} \frac{dx}{\sqrt{x}\,(1+x)}$

70. Let $I = \int_{0}^{1} x^{3/2}(1-x)^{1/2}\, dx$

We have $\beta(m, n) = \int_{0}^{1} x^{m-1}(1-x)^{n-1}\, dx$

Taking $m-1 = 3/2$, $n-1 = 1/2$ we get $m = 5/2$, $n = 3/2$

Hence $I = \beta\left(5/2, 3/2\right) = \frac{\Gamma(5/2)\,\Gamma(3/2)}{\Gamma(4)}$

$$I = \frac{\left[3/2 \cdot 1/2 \cdot \sqrt{\pi}\right]\left[1/2 \cdot \sqrt{\pi}\right]}{3!} = \frac{3\pi}{48} = \frac{\pi}{16}$$

Thus $\boldsymbol{I = \pi/16}$

---

71. Let $I = \int_{0}^{1} \sqrt{(1-x)/x}\;\, dx = \int_{0}^{1} x^{-1/2}(1-x)^{1/2}\, dx$

As above, taking $m-1 = -1/2$, $n-1 = 1/2$ we get $m = 1/2$, $n = 3/2$

Hence $I = \beta(1/2, 3/2) = \frac{\Gamma(1/2)\,\Gamma(3/2)}{\Gamma(2)} = \frac{\sqrt{\pi} \cdot 1/2 \cdot \sqrt{\pi}}{1} = \frac{\pi}{2}$

Thus $\boldsymbol{I = \pi/2}$

---

72. Let $I = \int_{0}^{2} (4-x^2)^{3/2}\, dx$

Put $x = 2\sin\theta \;\therefore\; dx = 2\cos\theta\, d\theta$

$$(4-x^2)^{3/2} = [4(1-\sin^2\theta)]^{3/2} = (2^2\cos^2\theta)^{3/2} = 8\cos^3\theta$$

If $x = 0$, $2\sin\theta = 0$ or $\sin\theta = 0 \Rightarrow \theta = 0$

$x = 2$, $2\sin\theta = 2$ or $\sin\theta = 1 \Rightarrow \theta = \pi/2$

Hence $I = \int_{\theta=0}^{\pi/2} 8\cos^3\theta \cdot 2\cos\theta\, d\theta = 16 \int_0^{\pi/2} \cos^4\theta\, d\theta$

Now $I = 16 \cdot \frac{1}{2}\beta\left(\frac{0+1}{2}, \frac{4+1}{2}\right) = 8\beta\left(\frac{1}{2}, \frac{5}{2}\right)$

$$I = 8 \cdot \frac{\Gamma(1/2)\,\Gamma(5/2)}{\Gamma(3)} = \frac{8 \cdot \sqrt{\pi} \cdot 3/2 \cdot 1/2 \cdot \sqrt{\pi}}{2!} = 3\pi$$

Thus $\boldsymbol{I = 3\pi}$

---

73. Let $I = \int_0^4 x^{3/2}(4-x)^{5/2}\, dx$

Put $x = 4\sin^2\theta \;\therefore\; dx = 8\sin\theta\cos\theta\, d\theta$. θ varies from 0 to $\pi/2$

$$\therefore \quad I = \int_0^{\pi/2} (2^2\sin^2\theta)^{3/2}(2^2\cos^2\theta)^{5/2} \cdot 8\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} 2^3\sin^3\theta \cdot 2^5\cos^5\theta \cdot 2^3\sin\theta\cos\theta\, d\theta$$

$$= 2^{11} \int_0^{\pi/2} \sin^4\theta\cos^6\theta\, d\theta$$

Now $I = 2^{11} \cdot \frac{1}{2}\beta\left(\frac{4+1}{2}, \frac{6+1}{2}\right) = 2^{10}\beta(5/2, 7/2)$

$$I = 2^{10}\frac{\Gamma(5/2)\,\Gamma(7/2)}{\Gamma(6)} = 2^{10} \cdot \frac{\Gamma(5/2) \cdot 5/2\,\Gamma(5/2)}{5!}$$

$$I = \frac{2^9}{4!} \cdot \left[\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}\right]^2 = \frac{2^5}{24} \cdot 9\pi = 12\pi$$

Thus $\boldsymbol{I = 12\pi}$

---

74. Let $I = \int_0^a x^4 \sqrt{a^2 - x^2}\, dx$

Put $x = a \sin\theta \quad \therefore \quad dx = a\cos\theta\, d\theta$ and θ varies from 0 to $\pi/2$

$$\therefore \qquad I = \int_0^{\pi/2} a^4 \sin^4\theta \cdot a\cos\theta \cdot a\cos\theta\, d\theta = a^6 \int_0^{\pi/2} \sin^4\theta \cos^2\theta\, d\theta$$

$$\text{Now} \qquad I = a^6 \cdot \frac{1}{2}\beta\left(5/2,\ 3/2\right) = \frac{a^6}{2} \cdot \frac{\Gamma(5/2)\cdot\Gamma(3/2)}{\Gamma(4)}$$

$$= \frac{a^6}{2} \cdot \frac{3/2\,\{\Gamma(3/2)\}^2}{3!} = \frac{a^6}{8} \cdot \left\{\frac{1}{2}\sqrt{\pi}\right\}^2 = \frac{\pi a^6}{32}$$

Thus $\boldsymbol{I = \pi a^6 / 32}$

---

75. Let $I = \int_0^2 \frac{x^2}{\sqrt{2-x}}\, dx$

Put $x = 2\sin^2\theta \quad \therefore \quad dx = 4\sin\theta\cos\theta\, d\theta$, θ varies from 0 to $\pi/2$

$$\therefore \qquad I = \int_0^{\pi/2} \frac{4\sin^4\theta \cdot 4\sin\theta\cos\theta}{\sqrt{2}\cos\theta}\, d\theta = 8\sqrt{2} \int_0^{\pi/2} \sin^5\theta \cos^0\theta\, d\theta$$

$$\text{Now} \qquad I = 8\sqrt{2} \cdot \frac{1}{2}\beta\left(3,\ 1/2\right) = 4\sqrt{2} \cdot \frac{\Gamma(3)\,\Gamma(1/2)}{\Gamma(7/2)}$$

$$I = 4\sqrt{2} \cdot \frac{2!\,\sqrt{\pi}}{5/2 \cdot 3/2 \cdot 1/2 \cdot \sqrt{\pi}} = \frac{64\sqrt{2}}{15}$$

Thus $\boldsymbol{I = 64\sqrt{2} / 15}$

---

76. Let $I = \int_{-1}^{1} \sqrt{1+x/1-x}\, dx$

Put $\quad x = \cos 2\theta \quad \therefore \quad dx = -2\sin 2\theta\, d\theta = -4\sin\theta\cos\theta\, d\theta$

If $\quad x = -1, \ \cos 2\theta = -1 \ \Rightarrow \ 2\theta = \pi$ or $\theta = \pi/2$

$\quad x = 1, \ \cos 2\theta = 1 \ \Rightarrow \ 2\theta = 0$ or $\theta = 0$

Also $\quad 1 + x = 1 + \cos 2\theta = 2\cos^2\theta \ ; \ 1 - x = 1 - \cos 2\theta = 2\sin^2\theta$

$$\therefore \quad I = \int_{\theta=\pi/2}^{0} \sqrt{2\cos^2\theta / 2\sin^2\theta} \cdot (-4\sin\theta\cos\theta)\, d\theta$$

$$= \int_{0}^{\pi/2} \frac{\cos\theta}{\sin\theta} \cdot 4\sin\theta\cos\theta = 4\int_{0}^{\pi/2} \sin^0\theta\cos^2\theta\, d\theta$$

$$I = 4\cdot\frac{1}{2}\beta\left(1/2\,,\ 3/2\right) = 2\frac{\Gamma(1/2)\Gamma(3/2)}{\Gamma(2)} = \frac{2\sqrt{\pi}\cdot\sqrt{\pi}/2}{1} = \pi$$

Thus $\boldsymbol{I = \pi}$

---

77. Let $I = \int_{0}^{\infty} \frac{dx}{1+x^4}$

Put $x^4 = \tan^2\theta$ *ie.,* $x = \tan^{1/2}\theta$; $\therefore$ $dx = 1/2 \cdot \tan^{-1/2}\theta\sec^2\theta\, d\theta$

θ varies from 0 to $\pi/2$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \frac{1/2\cdot\tan^{-1/2}\theta\sec^2\theta}{\sec^2\theta}\, d\theta = \frac{1}{2}\int_{0}^{\pi/2} \frac{\sin^{-1/2}\theta}{\cos^{-1/2}\theta}\, d\theta$$

$$I = \frac{1}{2}\int_{0}^{\pi/2} \sin^{-1/2}\theta\cos^{1/2}\theta\, d\theta = \frac{1}{2}\cdot\frac{1}{2}\beta\left(\frac{-1/2+1}{2}\,,\ \frac{1/2+1}{2}\right)$$

$$I = \frac{1}{4}\beta\left(1/4\,,\ 3/4\right) = \frac{1}{4}\frac{\Gamma(1/4)\Gamma(3/4)}{\Gamma(1)} = \frac{\pi\sqrt{2}}{4} = \frac{\pi}{2\sqrt{2}}$$

Thus $\boldsymbol{I = \pi / 2\sqrt{2}}$

---

78. Let $I = \int_{0}^{\infty} \frac{dx}{\sqrt{x}\,(1+x)}$

Put $x = \tan^2\theta$ $\therefore$ $dx = 2\tan\theta\sec^2\theta\, d\theta$, θ varies from 0 to $\pi/2$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \frac{2\tan\theta\sec^2\theta}{\tan\theta\sec^2\theta}\, d\theta = 2\int_{0}^{\pi/2} 1\, d\theta = 2\int_{0}^{\pi/2} \sin^0\theta\cos^0\theta\, d\theta$$

Now $I = 2\cdot\frac{1}{2}\beta\left(1/2\,,\ 1/2\right) = \frac{\Gamma(1/2)\cdot\Gamma(1/2)}{\Gamma(1)} = \sqrt{\pi}\cdot\sqrt{\pi} = \pi$

Thus $\boldsymbol{I = \pi}$

---

79. *Show that* $\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

>> Consider R.H.S and put $x = \tan^2\theta$.

$\therefore\; dx = 2\tan\theta\sec^2 d\theta$ and θ varies from 0 to $\pi/2$

$$\text{R.H.S} = \int_0^{\pi/2} \frac{(\tan^2\theta)^{m-1}\cdot 2\tan\theta\sec^2\theta}{(\sec^2\theta)^{m+n}}\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\tan^{2m-2+1}\theta\, d\theta}{\sec^{2m+2n-2}\theta} = 2\int_0^{\pi/2} \tan^{2m-1}\theta\cdot\cos^{2m+2n-2}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2m-1}\theta}{\cos^{2m-1}\theta}\cdot\cos^{2m+2n-2}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \beta(m, n) = \text{L.H.S.}$$

Thus $\boldsymbol{\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx}$

Note : $\boldsymbol{\beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx}$, since $\beta(m, n) = \beta(n, m)$

Remark : *This expression can also be regarded as an alternative form of the definition of the beta function and it will be quite useful.*

---

80. *Show that* $\beta(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$

Note : *We have to first establish the expression for* $\beta(m, n)$ *as in Problem-79.*

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\, dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$= I_1 + I_2 \;(say)$$

Consider $$I_2 = \int_1^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Put $x = 1/y \;\therefore\; dx = -(1/y^2)\,dy$. If $x = 1,\ y = 1\ ;\ x = \infty,\ y = 0$

$$\therefore \quad I_2 = \int_1^0 \frac{(1/y)^{m-1} \cdot -1/y^2\,dy}{(1+1/y)^{m+n}} = \int_0^1 \frac{1/y^{m+1}}{(\overline{y+1}/y)^{m+n}}\,dy$$

$$= \int_0^1 \frac{y^{m+n}/y^{m+1}}{(y+1)^{m+n}}\,dy = \int_0^1 \frac{y^{n-1}}{(y+1)^{m+n}}\,dy$$

The variable of integration is arbitrary in a definite integral, let us change y to x.

Thus $I_2 = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$ and $\beta(m, n) = I_1 + I_2$

$$\therefore \quad \beta(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Thus $$\boldsymbol{\beta(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\,dx}$$

---

81. *Given that* $\int_0^{\infty} \frac{x^{m-1}}{(1+x)}\,dx = \frac{\pi}{\sin m\pi}$ *show that* $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}$ *for* $0 < m < 1$

\>> We have $$\beta(m, n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

Taking $n = 1 - m$ we have,

$$\beta(m, 1-m) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)}\,dx = \frac{\pi}{\sin m\pi} \text{ by data.}$$

ie., $$\frac{\Gamma(m)\Gamma(1-m)}{\Gamma(1)} = \frac{\pi}{\sin m\pi}$$

ie., $$\boldsymbol{\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi}}$$

Corollary

Put $m = 1/4$: $\Gamma(1/4)\,\Gamma(3/4) = \dfrac{\pi}{\sin(\pi/4)} = \dfrac{\pi}{1/\sqrt{2}} = \pi\sqrt{2}$

Put $m = 1/3$: $\Gamma(1/3)\,\Gamma(2/3) = \dfrac{\pi}{\sin(\pi/3)} = \dfrac{\pi}{\sqrt{3}/2} = \dfrac{2\pi}{\sqrt{3}}$

Put $m = 1/2$: $\{\Gamma(1/2)\}^2 = \dfrac{\pi}{\sin(\pi/2)} = \pi \quad \therefore\ \Gamma(1/2) = \sqrt{\pi}$

Put $m = 1/6$: $\Gamma(1/6)\cdot\Gamma(5/6) = \dfrac{\pi}{\sin(\pi/6)} = \dfrac{\pi}{(1/2)} = 2\pi$

---

82. *Using* $\Gamma(p)\,\Gamma(1-p) = \pi/\sin p\pi$ *show that* $\displaystyle\int_0^\infty \frac{x^4}{1+x^6}\,dx = \frac{\pi}{3}$

\>> Let $\displaystyle I = \int_0^\infty \frac{x^4}{1+x^6}\,dx$

Put $x^6 = \tan^2\theta$ *ie.,* $x = \tan^{1/3}\theta \quad \therefore\ dx = \dfrac{1}{3}\tan^{-2/3}\theta\sec^2\theta\,d\theta$

θ varies from 0 to $\pi/2$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \frac{\tan^{4/3}\theta\cdot 1/3\cdot\tan^{-2/3}\theta\sec^2\theta}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2}\tan^{2/3}\theta\,d\theta = \frac{1}{3}\int_0^{\pi/2}\sin^{2/3}\theta\cos^{-2/3}\theta\,d\theta$$

Now $I = \dfrac{1}{3}\cdot\dfrac{1}{2}\beta\left(\dfrac{2/3+1}{2}, \dfrac{-2/3+1}{2}\right) = \dfrac{1}{6}\beta\left(\dfrac{5}{6}, \dfrac{1}{6}\right)$

Hence $I = \dfrac{1}{6}\dfrac{\Gamma(5/6)\,\Gamma(1/6)}{\Gamma(1)} = \dfrac{1}{6}\Gamma(5/6)\,\Gamma(1/6)$ ... (1)

But $\Gamma(p)\,\Gamma(1-p) = \pi/\sin p\pi$ by data. Put $p = 1/6$

$\therefore$ $\Gamma(1/6)\,\Gamma(5/6) = \pi/\sin(\pi/6) = \pi/(1/2) = 2\pi$

Now (1) becomes $I = 1/6\cdot 2\pi$

Thus $\boldsymbol{I = \pi/3}$

---

83. *Evaluate* $\int_0^2 (8-x^3)^{-1/3}\,dx$

>> Let $I = \int_0^2 (8-x^3)^{-1/3}\,dx$ and put $x^3 = 8\sin^2\theta$

ie., $x = 2\sin^{2/3}\theta \quad \therefore \quad dx = 2\cdot 2/3\cdot \sin^{-1/3}\theta\cos\theta\,d\theta$

If $x = 0,\ 8\sin^2\theta = 0 \Rightarrow \theta = 0$

$x = 2,\ 8\sin^2\theta = 8 \Rightarrow \theta = \pi/2$

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} (8\cos^2\theta)^{-1/3}\cdot\frac{4}{3}\sin^{-1/3}\theta\cos\theta\,d\theta$$

$$= \frac{1}{2}\cdot\frac{4}{3}\int_0^{\pi/2}\cos^{-2/3}\theta\sin^{-1/3}\theta\cos\theta\,d\theta$$

$$= \frac{2}{3}\int_0^{\pi/2}\sin^{-1/3}\theta\cos^{1/3}\theta\,d\theta$$

Now $I = \frac{2}{3}\cdot\frac{1}{2}\beta\left(\frac{-1/3+1}{2}, \frac{1/3+1}{2}\right) = \frac{1}{3}\beta\left(\frac{1}{3}, \frac{2}{3}\right)$

Hence $I = \frac{1}{3}\frac{\Gamma(1/3)\Gamma(2/3)}{\Gamma(1)} = \frac{1}{3}\cdot\frac{2\pi}{\sqrt{3}} = \frac{2\pi}{3\sqrt{3}}$

Thus $\mathbf{I = 2\pi/3\sqrt{3}}$

---

84. *Express* $\int_0^1 x^m(1-x^n)^p\,dx$ *interms of beta functions and hence evaluate* $\int_0^1 x^5(1-x^3)^{10}\,dx$

>> Let $I = \int_0^1 x^m(1-x^n)^p\,dx$

Put $x^n = \sin^2\theta$ or $x = \sin^{2/n}\theta \quad\therefore\quad dx = 2/n\cdot\sin^{(2/n)-1}\theta\cos\theta\,d\theta$

θ varies from 0 to $\pi/2$.

$$\therefore \quad I = \int_{\theta=0}^{\pi/2} \sin^{2m/n}\theta \cos^{2p}\theta \cdot 2/n \cdot \sin^{(2/n)-1}\theta \cos\theta \, d\theta$$

$$= \frac{2}{n}\int_0^{\pi/2} \sin^{(2m/n+2/n-1)}\theta \cos^{2p+1}\theta \, d\theta$$

Hence $I = \frac{2}{n}\cdot\frac{1}{2}\beta\left(\frac{2m/n+2/n}{2}, \frac{2p+2}{2}\right) = \frac{\mathbf{1}}{\boldsymbol{n}}\beta\left(\frac{\boldsymbol{m+1}}{\boldsymbol{n}}, \boldsymbol{p+1}\right)$

Also by putting $m = 5$, $n = 3$, $p = 10$ we get

$$\int_0^1 x^5(1-x^3)^{10}\,dx = \frac{1}{3}\beta(2, 11) = \frac{1}{3}\frac{\Gamma(2)\Gamma(11)}{\Gamma(13)}$$

Thus $\int_0^1 x^5(1-x^3)^{10}\,dx = \frac{1}{3}\cdot\frac{1!\,\Gamma(11)}{12\times 11\,\Gamma(11)} = \frac{\mathbf{1}}{\mathbf{396}}$

---

85. *Show that* $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$

>> Let $I_1 = \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx$

Put $x^4 = \sin^2\theta$ or $x = \sin^{1/2}\theta \quad \therefore \quad dx = 1/2 \cdot \sin^{-1/2}\theta\cos\theta\,d\theta$.

θ varies from 0 to $\pi/2$.

$$\therefore \quad I_1 = \int_{\theta=0}^{\pi/2} \frac{\sin\theta \cdot 1/2 \cdot \sin^{-1/2}\theta\cos\theta}{\cos\theta}\,d\theta = \frac{1}{2}\int_0^{\pi/2}\sin^{1/2}\theta\cos^0\theta\,d\theta$$

Now $I_1 = \frac{1}{2}\cdot\frac{1}{2}\beta\left(\frac{1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{4}\beta\left(3/4, 1/2\right) \quad \ldots(1)$

Let $I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Put $x^4 = \tan^2\theta$ or $x = \tan^{1/2}\theta$

$\therefore \quad dx = 1/2 \cdot \tan^{-1/2}\theta\sec^2\theta\,d\theta$

If $x = 0$ $\tan^2\theta = 0 \Rightarrow \theta = 0$; $x = 1$, $\tan^2\theta = 1 \Rightarrow \theta = \pi/4$

$$\therefore \quad I_2 = \int_{\theta=0}^{\pi/4} \frac{1/2 \cdot \tan^{-1/2}\theta \sec^2\theta}{\sec\theta}\, d\theta = \frac{1}{2}\int_0^{\pi/4} \frac{\sin^{-1/2}\theta}{\cos^{-1/2}\theta}\cdot\frac{1}{\cos\theta}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \sin^{-1/2}\theta \cos^{-1/2}\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin\theta\cos\theta}} = \frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\sin 2\theta}}$$

Now put $2\theta = \phi \;\therefore\; d\theta = d\phi/2$ and ϕ varies from 0 to $\pi/2$

Hence $I_2 = \dfrac{1}{\sqrt{2}}\displaystyle\int_{\phi=0}^{\pi/2} \frac{d\phi/2}{\sqrt{\sin\phi}} = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2}\phi\cos^0\phi\, d\phi$

$$I_2 = \frac{1}{2\sqrt{2}}\cdot\frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right) = \frac{1}{4\sqrt{2}}\beta(1/4, 1/2) \qquad \ldots (2)$$

From (1) and (2) $I_1 \times I_2 = \dfrac{1}{16\sqrt{2}}\beta(3/4, 1/2)\cdot\beta(1/4, 1/2)$

$$= \frac{1}{16\sqrt{2}}\frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(5/4)}\cdot\frac{\Gamma(1/4)\Gamma(1/2)}{\Gamma(3/4)}$$

$$= \frac{1}{16\sqrt{2}}\frac{\sqrt{\pi}}{1/4\cdot\Gamma(1/4)}\cdot\Gamma(1/4)\sqrt{\pi} = \frac{\pi}{4\sqrt{2}}$$

Thus $\mathbf{I_1 \times I_2 = \pi/4\sqrt{2}}$

---

86. *Show that* $\displaystyle\int_{-1}^{1}(1+x)^{p-1}(1-x)^{q-1}\,dx = 2^{p+q-1}\,\beta(p, q)$

>> Put $x = \cos 2\theta \;\therefore\; dx = -2\sin 2\theta\, d\theta = -4\sin\theta\cos\theta\, d\theta$

If $x = -1$, $\cos 2\theta = -1 \Rightarrow 2\theta = \pi$ or $\theta = \pi/2$

$x = 1$, $\cos 2\theta = 0 \Rightarrow 2\theta = 0$ or $\theta = 0$

L.H.S $= I = \displaystyle\int_{\pi/2}^{0}(1+\cos 2\theta)^{p-1}(1-\cos 2\theta)^{q-1}\cdot(-4\sin\theta\cos\theta)\,d\theta$

$$I = \int_0^{\pi/2} (2\cos^2\theta)^{p-1}(2\sin^2\theta)^{q-1} \cdot 2^2 \sin\theta\cos\theta\, d\theta$$

$$= 2^{p+q} \int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta\, d\theta = 2^{p+q} \cdot \frac{1}{2}\beta(q, p) = \text{R.H.S}$$

Thus $\boldsymbol{I = 2^{p+q-1}\beta(p, q)}$

---

87. *Show that* $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n} \frac{\Gamma(1/n)}{\Gamma(1/n+1/2)}$

>> Put $x^n = \sin^2\theta$ or $x = \sin^{2/n}\theta$

$\therefore$ $dx = 2/n \cdot \sin^{(2/n)-1}\theta\cos\theta\, d\theta$, θ varies from 0 to $\pi/2$

$$\text{L.H.S} = I = \int_{\theta=0}^{\pi/2} \frac{2/n \cdot \sin^{(2/n)-1}\theta\cos\theta}{\cos\theta}\, d\theta$$

$$I = \frac{2}{n}\int_0^{\pi/2} \sin^{(2/n)-1}\theta\cos^0\theta\, d\theta$$

Now $$I = \frac{2}{n}\cdot\frac{1}{2}\beta\left(\frac{2/n-1+1}{2}, \frac{0+1}{2}\right) = \frac{1}{n}\beta\left(\frac{1}{n}, \frac{1}{2}\right)$$

Thus $$I = \frac{1}{n}\frac{\Gamma(1/n)\Gamma(1/2)}{\Gamma(1/n+1/2)} = \frac{\sqrt{\pi}}{n}\frac{\Gamma(1/n)}{\Gamma(1/n+1/2)} = \textbf{R.H.S}$$

---

88. *Show that* $\displaystyle\int_0^1 \sqrt{1-x^4}\, dx = \frac{\sqrt{\pi}}{6}\frac{\Gamma(1/4)}{\Gamma(3/4)}$

>> Put $x^4 = \sin^2\theta$ or $x = \sin^{1/2}\theta$

$\therefore$ $dx = 1/2 \cdot \sin^{-1/2}\theta\cos\theta\, d\theta$ and θ varies from 0 to $\pi/2$

$$\text{L.H.S} = I = \int_{\theta=0}^{\pi/2} \cos\theta \cdot \frac{1}{2}\sin^{-1/2}\theta\cos\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta\cos^2\theta\, d\theta$$

Now $I = \frac{1}{2}\cdot\frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{2+1}{2}\right) = \frac{1}{4}\beta\left(\frac{1}{4}, \frac{3}{2}\right)$

Thus $I = \frac{1}{4}\frac{\Gamma(1/4)\,\Gamma(3/2)}{\Gamma(7/4)} = \frac{1}{4}\cdot\frac{\Gamma(1/4)\cdot\sqrt{\pi}/2}{3/4\,\Gamma(3/4)} = \frac{\sqrt{\pi}}{6}\frac{\Gamma(1/4)}{\Gamma(3/4)}$ = **R.H.S**

---

89. *Show that* $\int_0^1 x^{m-1}(1-x^2)^{n-1}\,dx = \frac{1}{2}\beta(m/2, n)$

>> Put $x = \sin\theta$ $\therefore$ $dx = \cos\theta\, d\theta$. θ varies from 0 to $\pi/2$

$$\text{L.H.S} = I = \int_{\theta=0}^{\pi/2} \sin^{m-1}\theta\cos^{2n-2}\theta\cdot\cos\theta\, d\theta$$

Thus L.H.S $= \int_0^{\pi/2} \sin^{2(m/2)-1}\theta\cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m/2, n)$ = **R.H.S**

---

90. *Show that* $\int_0^{\pi/2} \frac{d\theta}{\sqrt{2-\sin^2\theta}} = \frac{\sqrt{\pi}}{4}\frac{\Gamma(1/4)}{\Gamma(3/4)}$

>> Let $I = \int_0^{\pi/2}\frac{d\theta}{\sqrt{2-\sin^2\theta}}$

Put $\sin^2\theta = 2\sin^2\phi$ *ie.,* $\sin\theta = \sqrt{2}\sin\phi$ or $\theta = \sin^{-1}(\sqrt{2}\sin\phi)$

$\therefore$ $d\theta = \frac{1}{\sqrt{1-2\sin^2\phi}}\cdot\sqrt{2}\cos\phi\, d\phi = \frac{\sqrt{2}\cos\phi}{\sqrt{\cos 2\phi}}\, d\phi$

If $\theta = 0$, $\sin 0 = \sqrt{2}\sin\phi$ *ie.,* $\sqrt{2}\sin\phi = 0 \Rightarrow \phi = 0$

$\theta = \pi/2$, $\sin\pi/2 = \sqrt{2}\sin\phi$ *ie.,* $\sin\phi = 1/\sqrt{2} \Rightarrow \phi = \pi/4$

$$\therefore \quad I = \int_{\phi=0}^{\pi/4}\frac{\sqrt{2}\cos\phi/\sqrt{\cos 2\phi}}{\sqrt{2}\cos\phi}\, d\phi = \int_0^{\pi/4}\frac{1}{\sqrt{\cos 2\phi}}\, d\phi$$

Now put $2\phi = t$ $\therefore$ $d\phi = dt/2$. ϕ varies from 0 to $\pi/2$

Hence $I = \int_{t=0}^{\pi/2}\frac{dt/2}{\sqrt{\cos t}} = \frac{1}{2}\int_0^{\pi/2}\sin^0 t\,\cos^{-1/2} t\, dt$

Now $I = \frac{1}{2} \cdot \frac{1}{2} \beta \left(\frac{0+1}{2}, \frac{-1/2+1}{2} \right) = \frac{1}{4} \beta \left(\frac{1}{2}, \frac{1}{4} \right)$

Thus $I = \frac{1}{4} \frac{\Gamma(1/2)\Gamma(1/4)}{\Gamma(3/4)} = \frac{\sqrt{\pi}}{4} \frac{\Gamma(1/4)}{\Gamma(3/4)}$

---

91. *Show that* $\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$ *and hence show that*

$$\int_0^\infty \frac{x^3(1-x^5)}{(1+x)^{13}}\, dx = 0$$

\>> The first part is already worked out in the problem - 79.

Now let $I = \int_0^\infty \frac{x^3}{(1+x)^{13}}\, dx - \int_0^\infty \frac{x^8}{(1+x)^{13}}\, dx$

$$= \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+9}}\, dx - \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+4}}\, dx$$

ie., $I = \beta(4, 9) - \beta(9, 4)$. But $\beta(m, n) = \beta(n, m)$

Thus we have, $I = 0$

---

92. *Given that* $\cos(\pi/5) = \frac{\sqrt{5}+1}{4}$ *and* $\sin(\pi/10) = \frac{\sqrt{5}-1}{4}$ *show that* $\Gamma(1/5)\Gamma(2/5)\Gamma(3/5)\Gamma(4/5) = 4\pi^2/\sqrt{5}$

\>> We know that $\Gamma(m)\Gamma(1-m) = \pi/\sin m\pi$

Putting $m = 1/5$ and $m = 2/5$ in this result we have

$$\Gamma(1/5)\Gamma(4/5) = \pi/\sin(\pi/5) \text{ and } \Gamma(2/5)\Gamma(3/5) = \pi/\sin(2\pi/5)$$

Multiplying these two results we get,

$$\Gamma(1/5)\Gamma(2/5)\Gamma(3/5)\Gamma(4/5) = \frac{\pi^2}{\sin(\pi/5) \cdot \sin(2\pi/5)} \quad \ldots (1)$$

Now $\sin(\pi/5)\sin(2\pi/5)$

$$= \sin 36^\circ \cdot \sin 72^\circ = \frac{1}{2}\left[\cos(-36^\circ) - \cos(108^\circ)\right] = \frac{1}{2}\left[\cos 36^\circ - \cos 108^\circ\right]$$

But $\cos 108^\circ = \cos(90^\circ + 18^\circ) = -\sin 18^\circ$

$$\therefore \quad \sin(\pi/5)\sin(2\pi/5) = \frac{1}{2}\left[\cos(\pi/5) + \sin(\pi/10)\right]$$

$$= \frac{1}{2}\left[\frac{\sqrt{5}+1}{4} + \frac{\sqrt{5}-1}{4}\right] = \frac{\sqrt{5}}{4}$$

Hence $\sin(\pi/5)\sin(2\pi/5) = \sqrt{5}/4$ ... (2)

Thus by using (2) in (1) we get, $\Gamma(1/5)\Gamma(2/5)\Gamma(3/5)\Gamma(4/5) = 4\pi^2/\sqrt{5}$

EXERCISES

Evaluate the following **[1 *to* 6]**

1. $\int_0^\infty x^6 e^{-3x}\, dx$

2. $\int_0^\infty x^{-7/4} e^{-\sqrt{x}}\, dx$

3. $\int_0^\infty x^4 e^{-x^2}\, dx$

4. $\int_0^\infty a^{-bx^2}\, dx$

5. $\int_0^1 \sqrt{x} \log x\, dx$

6. $\dfrac{\Gamma(3)\Gamma(5/2)}{\Gamma(11/2)}$

7. Show that $\Gamma(-n+1/2) = \dfrac{(-1)^n 2^n \sqrt{\pi}}{1\cdot 3\cdot 5\cdots 2n-1}$

8. Prove that :

(i) $\int_0^\infty \cos(ax^{1/n})\, dx = \dfrac{1}{a^n}\Gamma(n+1)\cos(n\pi/2)$

(ii) $\int_0^\infty \sin(ax^{1/n})\, dx = \dfrac{1}{a^n}\Gamma(n+1)\sin(n\pi/2)$

9. Show that $\int_0^\infty \dfrac{e^{-x^2}}{\sqrt{x}}\, dx \times \int_0^\infty e^{-x^4} x^2\, dx = \dfrac{\pi}{4\sqrt{2}}$

10. Show that $\int_0^\infty e^{-a^2 x^2}\, dx = \dfrac{\sqrt{\pi}}{2a}$

Evaluate the following by expressing in terms of beta function **[11 *to* 14]**

11. $\int_0^{\pi/2} \sin^{1/2} x \cos^{3/2} x \, dx$

12. $\int_0^{\infty} \frac{x}{1+x^6} \, dx$

13. $\int_0^{2} x (8 - x^3)^{1/3} \, dx$

14. $\int_0^{\infty} \frac{x^2}{(1+x^4)^3} \, dx$

15. Find $\beta\,(1/2, 7/2)$

16. Find $\beta\,(5/6, 1/6)$

17. Show that $\int_{-a}^{+a} (a+x)^{m-1} (a-x)^{n-1} \, dx = (2a)^{m+n-1} \beta(m, n)$

18. Show that $\int_0^{\infty} \frac{x^{n-1}}{(x+a)^{m+n}} \, dx = \frac{1}{a^n} \beta(m, n)$

19. Show that $\int_0^{\infty} \frac{dx}{\sqrt{a^4 - x^4}} = \frac{[\Gamma(1/4)]^2}{4a\sqrt{2\pi}}$

20. Show that $\int_0^{\pi/2} \tan^p \theta \, d\theta = \frac{\pi}{2} \sec(p\pi/2)$

ANSWERS

1. 80/243

2. $8\sqrt{\pi}/3$

3. $3\sqrt{\pi}/8$

4. $\sqrt{\pi/b \log a}$

5. $-4/9$

6. 16/315

11. $\pi/4\sqrt{2}$

12. $\pi/3\sqrt{3}$

13. $16\pi/9\sqrt{3}$

14. $5\pi/64\sqrt{2}$

15. $5\pi/16$

16. 2π

PART B

Unit - VI

Vector Integration

6.1 Introduction

Basically *Vector* is a quantity having both magnitude and direction. Vector quantities like force, velocity, acceleration etc. have lot of reference in physical and engineering problems. We are familiar with vector algebra which gives an exposure to all the basic concepts related to vectors.

Differentiation and Integration are well acquainted topics in calculus. Further we have already discussed vector differentiation and with the knowledge of all these topics we discuss **Vector Integration.**

6.2 Vector line integral

Consider a curve C in space which consists of infinitesimally small line elements of length *dr*. Then the line integral of a vector $\vec{A}(x, y, z)$ along the curve C is defined to be the sum of the scalar products of $\vec{A}$ and $d\vec{r}$ and is represented by $\int_C \vec{A} \cdot d\vec{r}$

If C is a closed curve which do not intersect any where, the line integral around C is denoted by $\oint_C \vec{A} \cdot d\vec{r}$

If $\vec{F}$ is the force acted upon by a particle in displacing it along the curve C then $\int_C \vec{F} \cdot d\vec{r}$ represents the *total work done* by the force. It also represents the *circulation* of $\vec{F}$ about C where $\vec{F}$ represents the velocity of the fluid.

$\vec{F}$ is said to be *irrotational* if $\oint_C \vec{F} \cdot d\vec{r} = 0$

WORKED PROBLEMS

1. *If* $\vec{F} = xy\,i + yz\,j + zx\,k$, *evaluate* $\int_C \vec{F}\cdot d\vec{r}$ *where* C *is the curve represented by* $x = t,\ y = t^2,\ z = t^3,\ -1 \le t \le 1$.

>> We have $\vec{F} = xy\,i + yz\,j + zx\,k$ and $\vec{r} = x\,i + y\,j + z\,k$ will give $d\vec{r} = dx\,i + dy\,j + dz\,k$

$\therefore \quad \vec{F}\cdot d\vec{r} = xy\,dx + yz\,dy + zx\,dz$

Since $x = t,\ y = t^2,\ z = t^3$ by data, we obtain

$$dx = dt,\ dy = 2t\,dt,\ dz = 3t^2\,dt$$

Thus $\vec{F}\cdot d\vec{r} = t^3\,dt + t^5(2t)\,dt + t^4(3t^2)\,dt$

ie., $\vec{F}\cdot d\vec{r} = (t^3 + 2t^6 + 3t^6)\,dt = (t^3 + 5t^6)\,dt$

$$\therefore \quad \int_C F\cdot d\vec{r} = \int_{t=-1}^{1} (t^3 + 5t^6)\,dt$$

$$= \left[\frac{t^4}{4}\right]_{-1}^{1} + 5\left[\frac{t^7}{7}\right]_{-1}^{1}$$

Thus $\int_C \vec{F}\cdot d\vec{r} = \left(\frac{1}{4} - \frac{1}{4}\right) + 5\left(\frac{1}{7} + \frac{1}{7}\right) = \mathbf{\frac{10}{7}}$

---

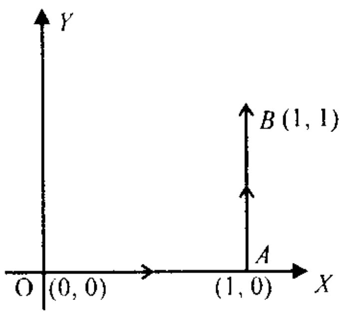
2. *Evaluate* $\int_C \vec{F}\cdot d\vec{r}$ *where* $\vec{F} = xy\,i + (x^2 + y^2)\,j$ *along*

(i) the path of the straight line from $(0, 0)$ *to* $(1, 0)$ *and then to* $(1, 1)$

(ii) the straight line joining the origin and $(1, 2)$

>> $$\int_C \vec{F}\cdot d\vec{r} = \int_C xy\,dx + (x^2 + y^2)\,dy \qquad \ldots (1)$$

(i) $$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} \qquad \ldots (2)$$



Along OA : $y = 0$

$\therefore \quad dy = 0$ and $0 \le x \le 1$

From (1) $$\int_{OA} \vec{F}\cdot d\vec{r} = 0 \qquad \ldots (3)$$

Along AB : $x = 1$ $\therefore$ $dx = 0$ and $0 \le y \le 1$. Again from (1)

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_{y=0}^{1} 0 + (1 + y^2)\, dy = \left[y + \frac{y^3}{3} \right]_{y=0}^{1} = 1 + \frac{1}{3} = \frac{4}{3} \qquad \ldots (4)$$

Using (3) and (4) in (2) we obtain $\int_C \vec{F} \cdot d\vec{r} = 0 + \frac{4}{3} = \frac{4}{3}$

(ii) C is the straight line joining $(0, 0)$ and $(1, 2)$

The equation of the line is given by $\frac{y-0}{x-0} = \frac{2-0}{1-0}$

ie., $y = 2x$ $\therefore$ $dy = 2\,dx$ and x varies from 0 to 1

Hence from (1) $\int_C \vec{F} \cdot \vec{dr} = \int_{x=0}^{1} x \cdot 2x\,dx + (x^2 + 4x^2)\, 2\,dx$

Thus $\int_C \vec{F} \cdot \vec{dr} = \int_{x=0}^{1} 12x^2\, dx = 12 \left[\frac{x^3}{3} \right]_0^1 = 4$

---

3. *If* $\vec{F} = (3x^2 + 6y)\, i - 14yz\, j + 20x z^2 k$, *evaluate* $\int \vec{F} \cdot d\vec{r}$ *from* $(0, 0, 0)$ *to* $(1, 1, 1)$ *along the curve given by* $x = t$, $y = t^2$, $z = t^3$

\>> $\vec{F} = (3x^2 + 6y)\, i - 14yz\, j + 20xz^2\, k$

$d\vec{r} = dx\, i + dy\, j + dz\, k$

$\therefore$ $\vec{F} \cdot d\vec{r} = (3x^2 + 6y)\, dx - 14yz\, dy + 20xz^2\, dz$

Since $x = t$, $y = t^2$, $z = t^3$ we obtain $dx = dt$, $dy = 2t\,dt$, $dz = 3t^2\,dt$

$\therefore$ $\vec{F} \cdot d\vec{r} = (3t^2 + 6t^2)\, dt - (14t^5)\, 2t\, dt + (20t^7)\, 3t^2\, dt$

ie., $\vec{F} \cdot d\vec{r} = (9t^2 - 28t^6 + 60t^9)\, dt$; $0 \le t \le 1$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=0}^{1} (9t^2 - 28t^6 + 60t^9)\, dt$$

Thus $\int_C \vec{F} \cdot d\vec{r} = \left[9\frac{t^3}{3} - 28\frac{t^7}{7} + 60\frac{t^{10}}{10} \right]_{t=0}^{1} = 3 - 4 + 6 = 5$

---

4. *If* $\vec{F} = x^2 i + xy\, j$ *evaluate* $\int_C \vec{F} \cdot d\vec{r}$ *from* $(0, 0)$ *to* $(1, 1)$ *along*

(i) the line $y = x$ *(ii) the parabola* $y = \sqrt{x}$

>> $\vec{F} \cdot d\vec{r} = x^2\, dx + xy\, dy$

(i) Along $y = x$: we have $0 \le x \le 1$ and $dy = dx$

$$\therefore \quad \int_C \vec{F} \cdot d\vec{r} = \int_{x=0}^{1} x^2\, dx + \int_{x=0}^{1} x^2\, dx = \int_0^1 2x^2\, dx = \left[\frac{2x^3}{3} \right]_0^1 = \mathbf{\frac{2}{3}}$$

(ii) Along $y = \sqrt{x}$: $y^2 = x$ and $2y\, dy = dx$, $0 \le y \le 1$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{y=0}^{1} 2y^5\, dy + \int_{y=0}^{1} y^3\, dy = \left[\frac{y^6}{3} \right]_0^1 + \left[\frac{y^4}{4} \right]_0^1 = \frac{1}{3} + \frac{1}{4} = \mathbf{\frac{7}{12}}$$

---

5. *Find the total work done by the force represented by* $\vec{F} = 3xy\, i - y\, j + 2zx\, k$ *in moving a particle round the circle* $x^2 + y^2 = 4$

>> Total work done $W = \int_C \vec{F} \cdot d\vec{r}$

$x^2 + y^2 = 4$ can be represented in the parametric form $x = 2\cos\theta$, $y = 2\sin\theta$ and $z = 0$. $0 \le \theta \le 2\pi$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int 3xy\, dx - y\, dy + 2\, zx\, dz$$

$$W = \int_{\theta=0}^{2\pi} 3\,(4\cos\theta\sin\theta)\,(-2\sin\theta)\, d\theta - \int_{\theta=0}^{2\pi} 4\sin\theta\cos\theta\, d\theta$$

$$= -24 \int_0^{2\pi} \sin^2\theta\cos\theta\, d\theta - 2\int_0^{2\pi} \sin 2\theta\, d\theta$$

$$= -24 \left[\frac{\sin^3\theta}{3} \right]_0^{2\pi} - 2\left[\frac{-\cos 2\theta}{2} \right]_0^{2\pi} = 0$$

Thus the total work done is 0

---

6. *If the acceleration of a particle at any time t is* $\vec{a} = 18\cos 3t\, i - 8\sin 2t\, j + 6t\, k$ *and* $\vec{v}$ *represents velocity and* $\vec{r}$ *represents displacement which are zero at* $t = 0$, *find* $\vec{r}$ *at time* $t > 0$

>> $\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = 18\cos 3t\, i - 8\sin 2t\, j + 6t\, k$

$\vec{r} = 0$ and $\vec{v} = \dfrac{d\vec{r}}{dt} = 0$ when $t = 0$, by data.

$\vec{a} = \dfrac{d^2\vec{r}}{dt^2} = \dfrac{d}{dt}\left(\dfrac{d\vec{r}}{dt}\right)$ and this implies that

$\dfrac{d\vec{r}}{dt} = \vec{v} = \int (18\cos 3t\, i - 8\sin 2t\, j + 6t\, k)\, dt$

ie., $\vec{v} = 6\sin 3t\, i + 4\cos 2t\, j + 3t^2 k + \vec{c}$ . . . (1)

Since $\vec{v} = 0$ at $t = 0$, (1) becomes $\vec{0} = 4j + \vec{c}$ $\therefore$ $\vec{c} = -4j$

Hence $\vec{v} = \dfrac{d\vec{r}}{dt} = 6\sin 3t\, i + 4\cos 2t\, j + 3t^2 k - 4j$

Integrating w.r.t t again we get

$\vec{r} = -2\cos 3t\, i + 2\sin 2t\, j + t^3 k - 4t\, j + \vec{d}$ . . . (2)

But $\vec{r} = 0$ at $t = 0$ and hence (2) becomes,

$\vec{0} = -2i + \vec{d}$ $\therefore$ $\vec{d} = 2i$.

Thus $\vec{r} = \mathbf{2(1 - \cos 3t)\, i + 2(\sin 2t - 2t)\, j + t^3 k}$, at any time t

---

7. *If* $\phi = 2xyz^2$, $\vec{F} = xy\, i - z\, j + x^2 k$ *and* C *is the curve* $x = t^2$, $y = 2t$, $z = t^3$ *from* $t = 0$ *to* $t = 1$, *evaluate the following line integrals.*

(i) $\int_C \phi\, d\vec{r}$ *(ii)* $\int_C \vec{F} \times d\vec{r}$

>> $d\vec{r} = dx\, i + dy\, j + dz\, k$

ie., $d\vec{r} = (2t\, i + 2j + 3t^2 k)\, dt$

$\phi = 2 \cdot t^2 \cdot 2t \cdot t^6 = 4t^9$

$\phi\, d\vec{r} = (8t^{10} i + 8t^9 j + 12t^{11} k)\, dt$

(i) $\int_C \phi \, d\vec{r} = \int_{t=0}^{1} (8t^{10} i + 8t^9 j + 12t^{11} k)\, dt$

$$= 8\left[\frac{t^{11}}{11}\right]_0^1 i + 8\left[\frac{t^{10}}{10}\right]_0^1 j + 12\left[\frac{t^{12}}{12}\right]_0^1 k$$

Thus $\int_C \phi \, d\vec{r} = \frac{8}{11} i + \frac{4}{5} j + k$

(ii) $\vec{F} = 2t^3 i - t^3 j + t^4 k$

$$\int_C \vec{F} \times d\vec{r} = \int_C \vec{F} \times \frac{d\vec{r}}{dt} dt$$

$$\vec{F} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} i & j & k \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix}$$

$$= i(-3t^5 - 2t^4) - j(6t^5 - 2t^5) + k(4t^3 + 2t^4)$$

$$\int_C \vec{F} \times d\vec{r} = \int_{t=0}^{1} (-3t^5 - 2t^4)\, i\, dt - \int_{t=0}^{1} 4t^5 j\, dt + \int_{t=0}^{1} (4t^3 + 2t^4)\, k\, dt$$

$$= \left[-\frac{t^6}{2} - \frac{2t^5}{5}\right]_0^1 i - 4\left[\frac{t^6}{6}\right]_0^1 j + \left[t^4 + \frac{2t^5}{5}\right]_0^1 k$$

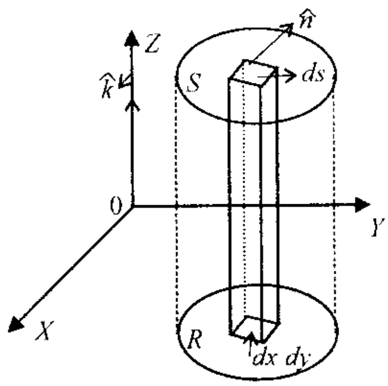
Thus $\int_C \vec{F} \times d\vec{r} = (-9/10)\, i - (2/3)\, j + (7/5)\, k$

6.3 Surface and Volume Integral

Surface integral

An integral evaluated over a surface is called a surface integral. Consider a surface S and a point P on it. Let $\vec{A}$ be a vector function of x, y, z defined and continuous over S.

If $\hat{n}$ is the unit outward normal to the surface S at P then the integral of the normal component of $\vec{A}$ at P (*ie.,* $\vec{A} \cdot \hat{n}$ *) over the surface* S *is called the* surface integral *written as* $\iint_S A \cdot \hat{n}\, ds$ *where* ds *is the small elemental area. To evaluate a surface integral we have to find the double integral over the orthogonal projection of the surface on one of the coordinate planes.*



Suppose R is the orthogonal projection of S on the xoy plane and $\hat{n}$ is the unit outward normal to S then it should be noted that $\hat{n}\cdot\hat{k}\,ds$ ($\hat{k}$ *being the unit vector along z-axis*) is the projection of the vectorial area element $\hat{n}\,ds$ on the xoy plane and this projection is equal to $dx\,dy$ which being the area element in the xoy plane. That is to say that $\hat{n}\cdot\hat{k}\,ds = dx\,dy$. Similarly we can argue to state that $\hat{n}\cdot\hat{j}\,ds = dz\,dx$ and $\hat{n}\cdot\hat{i}\,ds = dy\,dz$. All these three results hold good if we write

$$\hat{n}\,ds = dy\,dz\,i + dz\,dx\,j + dx\,dy\,k$$

Sometimes we also write, $\vec{ds} = \hat{n}\,ds = \Sigma\, dy\,dz\,i$

Volume integral

If V is the volume bounded by a surface and if $F(x, y, z)$ is a single valued function defined over V then the volume integral of $F(x, y, z)$ over V is given by $\iiint_V F\,dV$.

If the volume is divided into sub elements having sides dx, dy, dz then the volume integral is given by the triple integral $\iiint F(x, y, z)\,dx\,dy\,dz$ which can be evaluated by choosing appropriate limits for x, y, z.

We now proceed to state three integral theorems

6.4 Green's theorem in a plane

Statement : *If R is a closed region of the x - y plane bounded by a simple closed curve C and if M and N are two continuous functions of x, y having continuous first order partial derivatives in the region R then*

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

6.5 Stoke's theorem

Statement : *If S is a surface bounded by a simple closed curve C and if $\vec{F}$ is any continuously differentiable vector function then*

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,dS = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$$

6.6 Gauss divergence theorem

Statement : *If V is the volume bounded by a surface S and $\vec{F}$ is a continuously differentiable vector function then*

$$\iiint_V \text{div}\,\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,dS$$

where $\hat{n}$ is the positive unit vector outward drawn normal to S

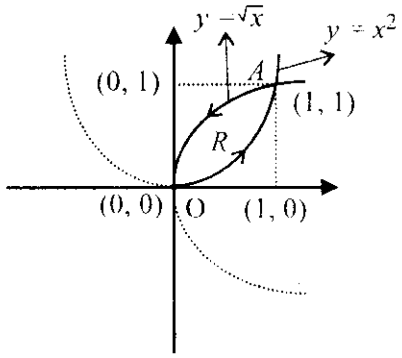
WORKED PROBLEMS

8. *Verify Green's theorem in a plane for* $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ *where C is the boundary of the region enclosed by* $y = \sqrt{x}$ *and* $y = x^2$

>> We shall find the points of intersection of the parabolas $y = \sqrt{x}$ and $y = x^2$

ie., $\sqrt{x} = x^2 \Rightarrow x = x^4$ or $x(x^3 - 1) = 0$

$\therefore$ $x = 0, 1$ and hence $y = 0, 1$. The points of intersection are $(0, 0)$ and $(1, 1)$



Let $M = 3x^2 - 8y^2,\ N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y\,;\quad \frac{\partial N}{\partial x} = -6y$$

We have Green's theorem in a plane

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

$$\text{L.H.S} = \oint_C M dx + N dy$$

$$= \int_{OA} M dx + N dy + \int_{AO} M dx + N dy = I_1 + I_2\ (say)$$

Along OA : $y = x^2,\ dy = 2x\,dx,\ x$ varies from 0 to 1

$$I_1 = \int_{x=0}^{1} (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx$$

$$= \int_{x=0}^{1} (3x^2 + 8x^3 - 20x^4)\,dx = \left[x^3 + 2x^4 - 4x^5 \right]_0^1 = -1$$

Along AO : $y = \sqrt{x}$ or $x = y^2 \Rightarrow dx = 2y\,dy,\ y$ varies from 1 to 0.

$$I_2 = \int_{y=1}^{0} (3y^4 - 8y^2)\,2y\,dy + (4y - 6y^3)\,dy$$

$$= \int_1^0 (4y - 22y^3 + 6y^5)\,dy = \left[2y^2 - \frac{11}{2}y^4 + y^6 \right]_1^0$$

$$= 0 - \left(2 - \frac{11}{2} + 1 \right) = \frac{5}{2}$$

Hence L.H.S $= I_1 + I_2 = -1 + 5/2 =$ **3/2**

Also, R.H.S $= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} (-6y + 16y)\, dy\, dx$$

$$= \int_{x=0}^{1} 10 \left[\frac{y^2}{2} \right]_{y=x^2}^{\sqrt{x}} dx$$

$$= 5 \int_{x=0}^{1} (x - x^4)\, dx$$

$$= 5 \left[\frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = 5 \left(\frac{1}{2} - \frac{1}{5} \right) = \frac{3}{2}$$

Thus we have **L.H.S = 3/2 = R.H.S** and hence the theorem is verified.

Note : *Suppose in the problem, we are asked to evauluate the line integral using Green's theorem we need to do only the R.H.S part of the theorem for obtaining the desired result.*

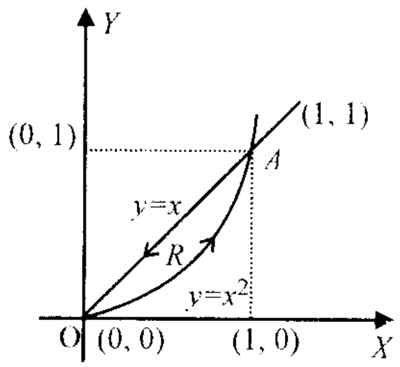
---

9. *Verify Green's theorem for* $\oint_C (xy + y^2)\, dx + x^2\, dy$ *where* C *is the closed curve of the region bounded by* $y = x$ *and* $y = x^2$

>> We shall find the points of intersection of $y = x$ and $y = x^2$.

Equating the R.H.S we have, $x = x^2$ or $x(1-x) = 0 \Rightarrow x = 0, 1$

$\therefore\ y = 0, 1$ and hence $(0, 0), (1, 1)$ are the points of intersection.



We have Green's theorem in a plane,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

$$\int_C (xy + y^2)\,dx + x^2\,dy = \int_{OA} (xy + y^2)\,dx + x^2\,dy + \int_{AO} (xy + y^2)\,dx + x^2\,dy = I_1 + I_2 \ (say)$$

Along OA we have $y = x^2 \ \therefore \ dy = 2x\,dx$ and x varies from 0 to 1

$$I_1 = \int_{x=0}^{1} (x \cdot x^2 + x^4)\,dx + x^2 \cdot 2x\,dx$$

$$= \int_{x=0}^{1} (3x^3 + x^4)\,dx = 3\left[\frac{x^4}{4} \right]_0^1 + \left[\frac{x^5}{5} \right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20}$$

Next, along AO we have $y = x \ \therefore \ dy = dx$ and x varies from 1 to 0.

$$I_2 = \int_{x=1}^{0} (x \cdot x + x^2)\,dx + x^2\,dx = \int_{x=1}^{0} 3x^2\,dx = \left[x^3 \right]_1^0 = -1$$

Hence L.H.S $= I_1 + I_2 = \frac{19}{20} - 1 = \frac{-1}{20}$

To evaluate the R.H.S we have $M = xy + y^2, \ N = x^2$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - (x + 2y) = x - 2y$$

R is the region bounded by $y = x^2$ and $y = x$.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy = \int_{x=0}^{1} \int_{y=x^2}^{x} (x - 2y)\,dy\,dx, \text{ from the figure.}$$

$$= \int_{x=0}^{1} \left[xy - y^2 \right]_{y=x^2}^{x} dx$$

$$= \int_{x=0}^{1} [\,(x^2 - x^2) - (x^3 - x^4)\,]\,dx$$

$$= \int_0^1 (x^4 - x^3)\,dx = \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1 = \frac{1}{5} - \frac{1}{4} = \frac{-1}{20}$$

Thus the theorem is verified.

---

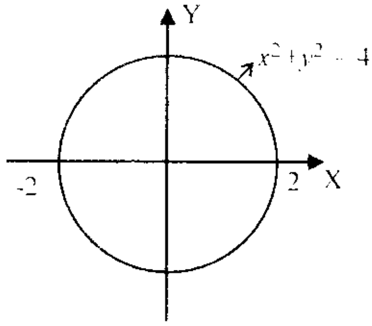
10. *Verify Green's theorem in the plane for* $\int_C (x^2+y^2)\,dx + 3x^2 y\,dy$ *where C is the circle* $x^2+y^2 = 4$ *traced in the positive sense.*

>> $\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$ *(Green's theorem)*

L.H.S = $\int_C (x^2+y^2)\,dx + 3x^2 y\,dy$ and the parametric equation of the given circle is

$x = 2\cos\theta,\ y = 2\sin\theta,\ 0 \le \theta \le 2\pi$

$$\text{L.H.S} = \int_{\theta=0}^{2\pi} 4(-2\sin\theta)\,d\theta + \int_{\theta=0}^{2\pi} 3\ (4\cos^2\theta)(2\sin\theta)(2\cos\theta)\,d\theta$$



$$= 8[\cos\theta]_0^{2\pi} + 48\int_0^{2\pi} \cos^3\theta \sin\theta\,d\theta$$

$$= 8(\cos 2\pi - \cos 0) - 48\left[\frac{\cos^4\theta}{4}\right]_0^{2\pi} = \mathbf{0}$$

$\because \quad \cos 2\pi = 1 = \cos 0$

Now if $M = x^2+y^2,\ N = 3x^2 y$ then $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = (6xy - 2y)$

$$\text{R.H.S} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2y(3x-1)\,dy\,dx$$

$$= \int_{x=-2}^{2} (3x-1)\left[y^2\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$\text{R.H.S} = \int_{x=-2}^{2} (3x-1)\left\{(4-x^2)-(4-x^2)\right\} = \mathbf{0}$$

Thus the theorem is verified.

11. *Employ Green's theorem in a plane to show that the area enclosed by a plane curve C is* $\frac{1}{2}\oint_C x\,dy - y\,dx$ *and hence find the area of the ellipse* $x^2/a^2 + y^2/b^2 = 1$

>> We have Green's theorem in a plane

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy \qquad \ldots (1)$$

Area enclosed by a plane curve C is $\iint_R dx\,dy$ ... (2)

Taking $N = \frac{x}{2}$, $M = \frac{-y}{2}$ we obtain $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{1}{2} - \left(-\frac{1}{2} \right) = 1$

Hence (1) becomes

$$\frac{1}{2} \int_C x\,dy - y\,dx = \iint_R dx\,dy = \text{Area}\,(A)$$

Now to find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ we have $x = a\cos\theta,\ y = b\sin\theta\ ;\ 0 \le \theta \le 2\pi$

$$dx = -a\sin\theta\,d\theta,\ dy = b\cos\theta\,d\theta$$

$$\therefore \qquad A = \frac{1}{2} \int_{\theta=0}^{2\pi} [\,a\cos\theta \cdot b\cos\theta - b\sin\theta\,(-a\sin\theta)\,]\,d\theta$$

ie.,
$$A = \frac{1}{2} \int_{\theta=0}^{2\pi} ab\,(\cos^2\theta + \sin^2\theta)\,d\theta = \frac{ab}{2}\left[\,\theta\,\right]_0^{2\pi} = \pi\,ab$$

Thus the required area $(A) = \pi\,ab$ sq. units.

---

12. *Find the area between the parabolas* $y^2 = 4x$ *and* $x^2 = 4y$ *with the help of Green's theorem in a plane.*

>> We have the area

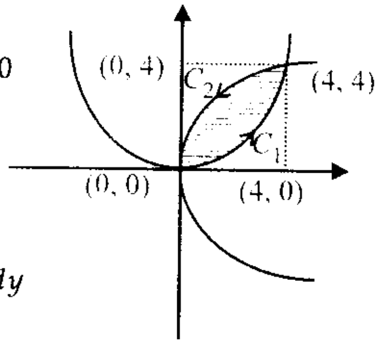
$$A = \iint dx\,dy = \frac{1}{2}\int_C x\,dy - y\,dx \qquad \textit{[Refer the previous problem]}$$

Let us find the points of intersection of $y^2 = 4x$ and $x^2 = 4y$

ie.,
$$\left(\frac{x^2}{4}\right)^2 = 4x \text{ or } x\,(x^3 - 64) = 0 \Rightarrow x = 0,\ x = 4 \quad \therefore \quad y = 0,\ 4$$

The points of intersection are $(0, 0)$ and $(4, 4)$.

C_1 is the curve $x^2 = 4y \;\therefore\; dy = \frac{x}{2}dx$ and $0 \le x \le 4$



C_2 is the curve $y^2 = 4x \;\therefore\; dx = \frac{y}{2}dy$ and $4 \le y \le 0$

$$\text{Now}\quad A = \frac{1}{2}\int_{C_1} x\,dy - y\,dx + \frac{1}{2}\int_{C_2} x\,dy - y\,dx$$

$$= \frac{1}{2}\int_{x=0}^{4}\left(x\cdot\frac{x}{2}-\frac{x^2}{4}\right)dx + \frac{1}{2}\int_{y=4}^{0}\left(\frac{y^2}{4}-\frac{y^2}{2}\right)dy$$

$$= \frac{1}{2}\int_{x=0}^{4}\frac{x^2}{4}dx - \frac{1}{2}\int_{y=0}^{4}-\frac{y^2}{4}dy$$

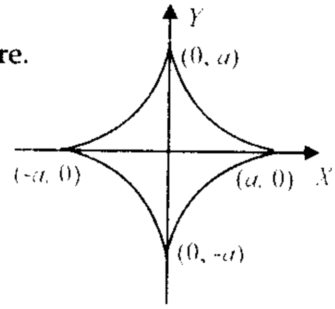
$$= \left[\frac{x^3}{24}\right]_0^4 + \left[\frac{y^3}{24}\right]_0^4 = \frac{64}{24}+\frac{64}{24} = \frac{128}{24} = \frac{16}{3}$$

Thus the required area = 16/3 sq. units.

---

13. *Find the area of the astroid : $x = a\cos^3\theta,\ y = a\sin^3\theta$ by employing Green's theorem.*

\>> The shape of the astroid is as shown in the figure.



Total area $A = 4$ (area in the first quadrant)

$$A = 4\times\frac{1}{2}\int_C x\,dy - y\,dx$$

$$A = 2\int_0^{\pi/2}[\,a\cos^3\theta\,(3a\sin^2\theta\cos\theta) - a\sin^3\theta\,(-3a\cos^2\theta\sin\theta)\,]\,d\theta$$

$$A = 6a^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,(\cos^2\theta+\sin^2\theta)\,d\theta$$

$$A = 6a^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta = 6a^2\cdot\frac{1\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}\quad\text{by reduction formula.}$$

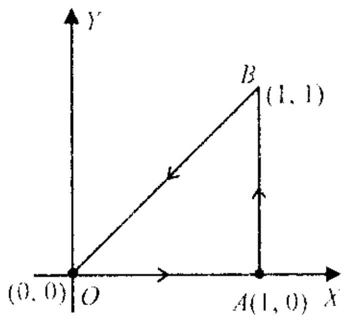
Thus the required area = $3\pi a^2/8$ sq. units.

---

14. *Evaluate* $\int_C (xy - x^2)\, dx + x^2 y\, dy$ *where C is the closed curve formed by* $y = 0$, $x = 1$ *and* $y = x$ *(a) directly as a line integral (b) by Green's theorem.*

(a) Let $M = xy - x^2$, $N = x^2 y$

$$\int_C Mdx + Ndy = \int_{OA} Mdx + Ndy + \int_{AB} Mdx + Ndy + \int_{BO} Mdx + Ndy$$



(i) Along $OA : y = 0 \Rightarrow dy = 0$ and $0 \le x \le 1$

(ii) Along $AB : x = 1 \Rightarrow dx = 0$ and $0 \le y \le 1$

(iii) Along $BO : y = x \Rightarrow dy = dx$ and $1 \le x \le 0$

$$\therefore \quad \int_C Mdx + Ndy = \int_{x=0}^{1} -x^2\, dx + \int_{y=0}^{1} y\, dy + \int_{x=1}^{0} x^3\, dx$$

$$= -\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{y^2}{2}\right]_0^1 + \left[\frac{x^4}{4}\right]_1^0 = \frac{-1}{3} + \frac{1}{2} - \frac{1}{4} = \frac{-1}{12}$$

Thus $\quad \int_C (xy - x^2)\, dx + x^2 y\, dy = \frac{-1}{12}$

(b) We have Green's theorem $\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$$\text{R.H.S} = \iint_R (2xy - x)\, dx dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} (2xy - x)\, dy dx, \text{ from the figure.}$$

$$= \int_{x=0}^{1} [\, xy^2 - xy\,]_{y=0}^{x}\, dx = \int_{x=0}^{1} (x^3 - x^2)\, dx$$

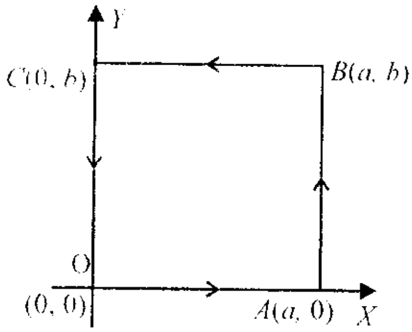
Thus $\quad$ **R.H.S** $= \left[\frac{x^4}{4} - \frac{x^3}{3}\right]_0^1 = \frac{1}{4} - \frac{1}{3} = \frac{-1}{12}$

---

15. *Verify Stoke's theorem for the vector $\vec{F} = (x^2 + y^2)\, i - 2xy\, j$ taken round the rectangle bounded by $x = 0,\ x = a,\ y = 0,\ y = b$*

>> We have Stoke's theorem : $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds$

$$\vec{F} \cdot d\vec{r} = (x^2 + y^2)\, dx - 2xy\, dy$$



$$\oint_C \vec{F} \cdot d\vec{r} = \int_{OA} \vec{F} \cdot d\vec{r} + \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot d\vec{r} + \int_{CO} \vec{F} \cdot d\vec{r}$$

$$\oint_C \vec{F} \cdot d\vec{r} = I_1 + I_2 + I_3 + I_4 \ (say)$$

(i) Along OA : $y = 0 \Rightarrow dy = 0$ and $0 \le x \le a$

$$\therefore \quad I_1 = \int_{x=0}^{a} x^2\, dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

(ii) Along AB : $x = a \Rightarrow dx = 0$ and $0 \le y \le b$

$$\therefore \quad I_2 = \int_{y=0}^{b} -2ay\,dy = \left[-ay^2\right]_0^b = -ab^2$$

(iii) Along BC : $y = b \Rightarrow dy = 0$ and $a \le x \le 0$

$$\therefore \quad I_3 = \int_{x=a}^{0} (x^2 + b^2)\, dx = \left[\frac{x^3}{3} + b^2 x\right]_a^0 = -\frac{a^3}{3} - ab^2$$

(iv) Along CO : $x = 0 \Rightarrow dx = 0$ and $b \le y \le 0$

$$\therefore \quad I_4 = \int_{y=b}^{0} 0\ dy = 0$$

Hence $\oint_C \vec{F} \cdot d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = \mathbf{-2ab^2}$ . . . (1)

Now, $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = k(-2y-2y) = -4y\,k$

$(\nabla \times \vec{F}) \cdot \hat{n}\,ds = (-4y\,k) \cdot (dy\,dz\,i + dz\,dx\,j + dx\,dy\,k) = -4y\,dx\,dy$

$\therefore \quad \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,ds = \iint_R -4y\,dx\,dy$

$$= -4 \int_{x=0}^{a} \int_{y=0}^{b} y\,dy\,dx$$

$$= -4 \int_{x=0}^{a} \left[\frac{y^2}{2}\right]_0^b dx = -2b^2\,[x]_0^a = -2ab^2$$

Hence we have, $\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,ds = -2ab^2$ ... (2)

Thus from (1) and (2) we conclude that the theorem is verified.

---

16. *Verify Stoke's theorem for $\vec{F} = y\,i + z\,j + x\,k$ where S is the upper half of the sphere $x^2+y^2+z^2 = 1$ and C is its boundary.*

>> We have Stoke's theorem $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\,ds$, C is the circle in the $x-y$ plane whose centre is the origin and radius equal to unity.

That is $x^2+y^2 = 1,\ z = 0$ or $x = \cos\theta,\ y = \sin\theta,\ 0 \le \theta \le 2\pi$

$$\int_C \vec{F} \cdot dr = \int_C y\,dx + z\,dy + x\,dz$$

$$= \int_{\theta=0}^{2\pi} \sin\theta \cdot (-\sin\theta)\,d\theta \qquad (\because\ z = 0,\ dz = 0)$$

$$= \int_{\theta=0}^{2\pi} -\sin^2\theta\,d\theta = \int_{\theta=0}^{2\pi} \frac{\cos 2\theta - 1}{2}\,d\theta = \frac{1}{2}\left[\frac{\sin 2\theta}{2} - \theta\right]_0^{2\pi} = -\pi$$

Hence we have, L.H.S $= \int_C \vec{F} \cdot d\vec{r} = -\pi$

Now, $\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -i - j - k$

$$\iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds = \iint_S \text{curl}\,\vec{F} \cdot \vec{ds} \text{ where we have,}$$

$$\vec{ds} = dydz\,i + dzdx\,j + dxdy\,k$$

$$\iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds = \iint_S -dxdy \quad (\because z = 0).$$

But $\iint dx\,dy$ represents the area of the circle $x^2 + y^2 = 1$ which is equal to π since $r = 1$.

Hence R.H.S $= \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds = -\pi$ . . . (2)

Thus from (1) and (2) we conclude that the theorem is verifed.

---

17. *Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\,i - 2xy\,j$ taken round the rectangle bounded by the lines $x = \pm a$, $y = 0$ and $y = b$*

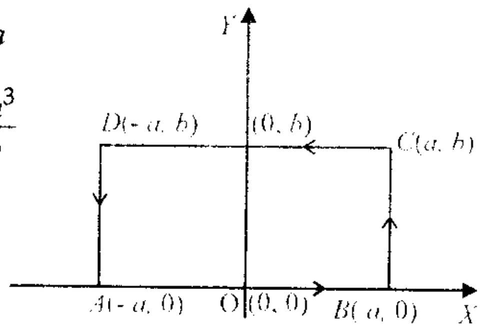
>> $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds$ [*Stoke's theorem*]

$$\int_C \vec{F} \cdot d\vec{r} = \int_{AB} \vec{F} \cdot d\vec{r} + \int_{BC} \vec{F} \cdot dr + \int_{CD} \vec{F} \cdot d\vec{r} + \int_{DA} \vec{F} \cdot d\vec{r} = I_1 + I_2 + I_3 + I_4$$

$$\vec{F} \cdot d\vec{r} = (x^2 + y^2)\,dx - 2xy\,dy$$

(i) Along AB : $y = 0 \Rightarrow dy = 0$; $-a \le x \le a$

$$I_1 = \int_{-a}^{a} x^2\, dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{a^3}{3} - \left(\frac{-a^3}{3}\right) = \frac{2a^3}{3}$$



(ii) Along BC : $x = a \Rightarrow dx = 0$; $0 \le y \le b$

$$I_2 = \int_{y=0}^{b} -2ay\,dy = [-a y^2]_0^b = -ab^2$$

(iii) Along CD : $y = b \Rightarrow dy = 0$; $a \le x \le -a$

$$I_3 = \int_{x=a}^{-a} (x^2 + b^2)\,dx = \left[\frac{x^3}{3} + b^2 x\right]_{x=a}^{-a}$$

$$= \left(\frac{-a^3}{3} - ab^2\right) - \left(\frac{a^3}{3} + ab^2\right) = \frac{-2a^3}{3} - 2ab^2$$

(iv) Along DA : $x = -a \Rightarrow dx = 0$; $b \le y \le 0$

$$I_4 = \int_{y=b}^{0} 2a\,y\,dy = [a y^2]_b^0 = -ab^2$$

Hence $\int_C \vec{F} \cdot d\vec{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 = \mathbf{-4ab^2}$ . . . (1)

Now $\quad \text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 + y^2) & -2xy & 0 \end{vmatrix} = -4y\,k$

$$\vec{ds} = \hat{n}\,ds = dy\,dz\,i + dz\,dx\,j + dx\,dy\,k$$

$\therefore \quad \text{curl}\,\vec{F} \cdot \hat{n}\,ds = -4y\,dx\,dy$

$$\iint_S \text{curl}\,\vec{F} \cdot \hat{n}\,ds = -4 \int_{x=-a}^{a} \int_{y=0}^{b} y\,dy\,dx$$

$$= -4 \int_{x=-a}^{a} \left[\frac{y^2}{2}\right]_0^b dx = -2 \int_{x=-a}^{a} b^2\,dx$$

$$= -2b^2 [x]_{-a}^{a} = -2b^2 (2a) = -4ab^2$$

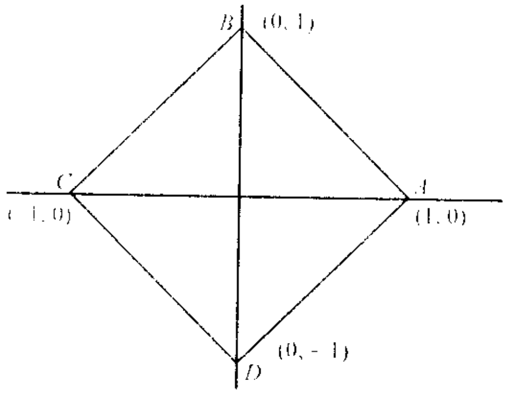
Thus $\quad \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\,ds = \mathbf{-4ab^2}$ . . . (2)

Thus from (1) and (2) we conclude that the theorem is verified.

22. *Evaluate* $\int_C xy\,dx + xy^2\,dy$ *by Stoke's theorem where C is the square in the x-y plane with vertices* (1, 0) (– 1, 0) (0, 1) (0, – 1)

>> We have Stoke's theorem,

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot \vec{ds}$$



From the given integral it is evident that

$$\vec{F} = xy\,\hat{i} + xy^2\,\hat{j}$$

since $\vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

Hence $\int_C xy\,dx + xy^2\,dy = \int_C \vec{F}\cdot\vec{dr}$

which is to be evaluated by applying Stoke's theorem.

$$\text{Now, Curl } \vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix}$$

ie., Curl $\vec{F} = (y^2 - x)\,\hat{k}$, on expanding the determinant.

Further $\vec{ds} = dydz\,\hat{i} + dzdx\,\hat{j} + dxdy\,\hat{k}$

$$\therefore \quad \iint_S \text{Curl } \vec{F}\cdot\vec{ds} = \iint (y^2 - x)\,dxdy$$

It can be clearly seen from the figure that $-1 \le x \le 1$ and $-1 \le y \le 1$

$$\text{Now } \iint_S \text{Curl } \vec{F}\cdot\vec{ds} = \int_{x=-1}^{1}\int_{y=-1}^{1} (y^2 - x)\,dy\,dx$$

$$= \int_{x=-1}^{1} \left[\frac{y^3}{3} - xy\right]_{y=-1}^{1} dx$$

$$= \int_{x=-1}^{1} \left[\left(\frac{1}{3} + \frac{1}{3} \right) - x(1+1) \right] dx$$

$$= \int_{x=-1}^{1} \left[\frac{2}{3} - 2x \right] dx$$

$$= \left[\frac{2}{3} x - x^2 \right]_{-1}^{1} = \frac{2}{3}(1+1) - (1-1) = \frac{4}{3}$$

Thus $\int_C xy\, dx + xy^2\, dy = \frac{4}{3}$

---

19. *Verify Stoke's theorem for $\vec{F} = (2x - y)\, i - yz^2\, j - y^2 z\, k$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$, C is its boundary.*

>> $\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n}\, ds$ *(Stoke's theorem)*

C is the circle : $x^2 + y^2 = 1,\ z = 0$

$$\vec{F} \cdot d\vec{r} = (2x - y)\, dx - yz^2\, dy - y^2 z\, dz = (2x - y)\, dx \quad (\because z = 0)$$

Taking $x = \cos\theta,\ y = \sin\theta$, where $0 \le \theta \le 2\pi$

$$\text{L.H.S} = \int_C \vec{F} \cdot d\vec{r} = \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} \left\{ -\sin 2\theta + \frac{1}{2}(1 - \cos 2\theta) \right\} d\theta$$

$$= \left[\frac{\cos 2\theta}{2} + \frac{\theta}{2} - \frac{\sin 2\theta}{4} \right]_0^{2\pi} = \left(\frac{1}{2} - \frac{1}{2} \right) + (\pi - 0) - 0 = \pi$$

Hence $\int_C \vec{F} \cdot d\vec{r} = \pi$ ... (1)

Also, $\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$

$$= i(-2yz+2yz) - j(0) + k(0+1) = k$$

$\therefore \quad \vec{ds} = \hat{n}\,ds = dy\,dz\,i + dz\,dx\,j + dx\,dy\,k$

Hence R.H.S $= \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\,ds = \iint dx\,dy = \pi$ . . . (2)

$\because \iint dx\,dy$ represents the area of the circle $x^2 + y^2 = 1$ which is π.

Thus from (1) and (2) we conclude that the theorem is verified.

---

20. *If* $\vec{F} = (2x^2 - 3z)\,i - 2xyj - 4x\,k$ *evaluate* $\iiint_V \nabla \cdot \vec{F}\,dV$ *where V is the region bounded by the planes* $x = 0,\ y = 0,\ z = 0$ *and* $2x + 2y + z = 4$

\>> $\quad \nabla \cdot \vec{F} = \dfrac{\partial}{\partial x}(2x^2 - 3z) + \dfrac{\partial}{\partial y}(-2xy) + \dfrac{\partial}{\partial z}(-4x) = 2x$

Here z varies from 0 to $4 - 2x - 2y$

When $z = 0$ we get $2x + 2y = 4$ or $x + y = 2$. y varies from 0 to $2 - x$.

$z = 0,\ y = 0$ will give us $2x = 4$ or $x = 2$. x varies from 0 to 2 .

$$\therefore \quad \iiint_V \nabla \cdot \vec{F}\,dV = \int_{x=0}^{2} \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 2x\,dz\,dy\,dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{2-x} 2x \Big[\, z \,\Big]_{z=0}^{4-2x-2y} dy\,dx$$

$$= 2\int_{x=0}^{2} \int_{y=0}^{2-x} x\,(4 - 2x - 2y)\,dy\,dx$$

$$\iiint_V \nabla \cdot \vec{F}\, dV = 2 \int_{x=0}^{2} \left[4xy - 2x^2 y - xy^2 \right]_{y=0}^{2-x} dx$$

$$= 2 \int_{x=0}^{2} [\, 4x(2-x) - 2x^2(2-x) - x(2-x)^2 \,]\, dx$$

$$= 2 \int_{x=0}^{2} (x^3 - 4x^2 + 4x)\, dx$$

$$\iiint_V \nabla \cdot \vec{F}\, dV = 2\left[\frac{x^4}{4} - \frac{4x^3}{3} + 2x^2 \right]_{x=0}^{2} = 2\left(4 - \frac{32}{3} + 8 \right) = \frac{8}{3}$$

---

21. *Verify divergence theorem for the vector function* $\vec{F} = (x^2 - yz)\, i + (y^2 - zx)\, j + (z^2 - xy)\, k$ *taken over the rectangular parallelopiped* $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$

\>> We have divergence theorem : $\iiint_V \text{div}\, \vec{F}\, dV = \iint_S \vec{F} \cdot \hat{n}\, ds$

$$\text{div}\, \vec{F} = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= 2(x + y + z)$$

$$\text{L.H.S} = \iiint_V \text{div}\, \vec{F}\, dV$$

$$= \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} 2(x + y + z)\, dz\, dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b} \left[2xz + 2yz + z^2 \right]_{z=0}^{c} dy\, dx$$

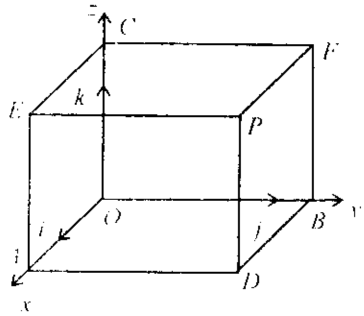
$$= \int_{x=0}^{a} \int_{y=0}^{b} (2cx + 2cy + c^2)\, dy\, dx = \int_{x=0}^{a} \left[2cxy + cy^2 + c^2 y \right]_{y=0}^{b} dx$$

$$\text{L.H.S} = \int_{x=0}^{a} (2bcx + cb^2 + c^2 b)\, dx = [\, bcx^2 + b^2 cx + bc^2 x \,]_{x=0}^{a}$$

$$\text{L.H.S} = a^2 bc + ab^2 c + abc^2$$

Hence L.H.S $= abc(a+b+c)$ ...(1)

Now R.H.S $= \iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S \vec{F}\cdot\vec{ds}$ is to be evaluated over the 6 faces of the rectangular parallelopiped namely



S_1 : $OADB$, S_2 : $OCEA$, S_3 : $OBFC$

S_4 : $CEPF$, S_5 : $BFPD$, S_6 : $ADPE$

The unit outward normals to these faces ($\hat{n}$) are respectively :

$$-k,\ -j,\ -i,\ k,\ j,\ i$$

(i) $\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{a}\int_{y=0}^{b} xy\;dy\;dx \qquad (z=0,\ \hat{n}=-k)$

$$= \int_{x=0}^{a}\left[\frac{xy^2}{2}\right]_{y=0}^{b} dx = \frac{b^2}{2}\left[\frac{x^2}{2}\right]_0^a = \frac{a^2b^2}{4} \qquad \ldots(1)$$

(ii) $\iint_{S_2} \vec{F}\cdot\hat{n}\;ds = \int_{x=0}^{a}\int_{z=0}^{c} zx\;dz\;dx = \frac{c^2a^2}{4}$ ...(2)

$(y=0,\ \hat{n}=-j)$

(iii) $\iint_{S_3} \vec{F}\cdot\hat{n}\;ds = \int_{y=0}^{b}\int_{z=0}^{c} yz\;dz\;dy = \frac{b^2c^2}{4}$ ...(3)

$(x=0,\ \hat{n}=-i)$

(iv) $\iint_{S_4} \vec{F}\cdot\hat{n}\;ds = \int_{x=0}^{a}\int_{y=0}^{b} (c^2-xy)\;dy\;dx$

$(\hat{n}=k,\ z=c)$

$$= \int_{x=0}^{a}\left[c^2y - x\frac{y^2}{2}\right]_{y=0}^{b} dx$$

$$= \int_{x=0}^{a}\left(c^2b-\frac{b^2}{2}x\right)dx = c^2b\,[x]_0^a - \frac{b^2}{2}\left[\frac{x^2}{2}\right]_0^a$$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = abc^2 - \frac{a^2 b^2}{4} \qquad \ldots(4)$$

(v) $$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{a}\int_{z=0}^{c} (b^2 - zx)\,dz\,dx = acb^2 - \frac{a^2 c^2}{4} \qquad \ldots(5)$$

$(\hat{n} = j,\ y = b)$

(vi) $$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{b}\int_{z=0}^{c} (a^2 - yz)\,dz\,dy = a^2 bc - \frac{b^2 c^2}{4} \qquad \ldots(6)$$

$(\hat{n} = i,\ x = a)$

Adding all the results from (1) to (6) we obtain

$$\iint_S F\cdot\hat{n}\,ds = abc^2 + acb^2 + a^2 bc = \mathbf{abc(a+b+c)} \qquad \ldots(\text{II})$$

Thus from (I) and (II) we conclude that the theorem is verified.

Note : A Question Format

Employ Gauss divergence theorem to evaluate $\iint_S \vec{F}\cdot\vec{n}\,ds$ *where* $\vec{F} = (x^2 - yz)\,i + (y^2 - zx)\,j + (z^2 - xy)\,k$ *taken over the rectangular parallelopiped,* $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$

>> We have $\iiint_V \text{div}\,\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,ds$

Here we need to work out only the L.H.S to obtain the result $\mathbf{abc(a+b+c)}$

---

22. *Use divergence theorem to evaluate* $\iint_S \vec{F}\cdot\hat{n}\,ds$ *over the entire surface of the region above xy plane bounded by the cone* $z^2 = x^2 + y^2$ *the plane* $z = 4$ *where* $\vec{F} = 4xz\,i + xyz^2\,j + 3z\,k$

>> We have $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dV$

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = 4z + xz^2 + 3$$

$$\iiint_V \text{div}\,\vec{F}\,dV = \iiint_V (4z + xz^2 + 3)\,dx\,dy\,dz \text{ with suitable limits.}$$

Putting $z = 4$ in $z^2 = x^2 + y^2$ we get $x^2 + y^2 = 16$

Hence y varies from $-\sqrt{16 - x^2}$ to $\sqrt{16 - x^2}$

If $y = 0$: $x^2 = 16$ and x varies for -4 to 4, z varies from 0 to 4

$$\iiint_V \operatorname{div} \vec{F}\, dV = \int_{z=0}^{4}\int_{x=-4}^{4}\int_{y=-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (4z + xz^2 + 3)\, dy\, dx\, dz$$

$$= \int_{z=0}^{4}\int_{x=-4}^{4} (4z + xz^2 + 3)\Big[\, y \,\Big]_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\, dx\, dz$$

$$= \int_{z=0}^{4}\int_{x=-4}^{4} (4z + xz^2 + 3)\, 2\sqrt{16-x^2}\, dx\, dz$$

$$= 8\int_{z=0}^{4}\int_{x=-4}^{4} z\sqrt{16-x^2}\, dx\, dz + 2\int_{z=0}^{4}\int_{x=-4}^{4} z^2 x\sqrt{16-x^2}\, dx\, dz$$

$$+ 6\int_{z=0}^{4}\int_{x=-4}^{4} \sqrt{16-x^2}\, dx\, dz$$

$$= 8\int_{z=0}^{4} z\cdot 2\int_{0}^{4}\sqrt{16-x^2}\, dx\, dz + 0 + 6\int_{z=0}^{4} 2\int_{x=0}^{4}\sqrt{16-x^2}\, dx\, dz$$

(Second term is zero since $x\sqrt{16-x^2}$ is an odd function)

$$= 16\int_{z=0}^{4} z\left[\frac{x\sqrt{16-x^2}}{2} + 8\sin^{-1}\left(\frac{x}{4}\right)\right]_{x=0}^{4} dz + 12\int_{z=0}^{4}\left[\frac{x\sqrt{16-x^2}}{2} + 8\sin^{-1}\left(\frac{x}{4}\right)\right]_{x=0}^{4} dz$$

$$= 16\int_{z=0}^{4} z\,[\,0 + 8\,(\pi/2 - 0)\,]\, dz + 12\int_{z=0}^{4}[\,0 + 8\,(\pi/2 - 0\,]\, dz$$

$$= 64\pi\left[\frac{z^2}{2}\right]_{z=0}^{4} + 48\pi\,[\,z\,]_0^4 = 512\pi + 192\pi = 704\pi$$

Thus $\displaystyle\iint_S \vec{F}\cdot\hat{n}\, ds = 704\pi$

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23. *Evaluate* $\iint_S (yz\,i + zx\,j + xy\,k)\cdot\hat{n}\,ds$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *in the first octant.*

>> The given surface is $x^2 + y^2 + z^2 = a^2$

We know that $\nabla\phi$ is a vector normal to the surface $\phi(x, y, z) = c$. Taking $\phi(x, y, z) = x^2 + y^2 + z^2$

$$\nabla\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k = 2x\,i + 2y\,j + 2z\,k$$

$\therefore$ unit vector normal $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2(xi + yj + zk)}{\sqrt{2^2(x^2 + y^2 + z^2)}}$

ie., $\hat{n} = \dfrac{xi + yj + zk}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{xi + yj + zk}{a} \quad \because x^2 + y^2 + z^2 = a^2$

Also if $\vec{F} = yz\,i + zx\,j + xy\,k$,

$$\vec{F}\cdot\hat{n} = \frac{1}{a}(xyz + yzx + zxy) = \frac{3xyz}{a} \qquad \ldots(1)$$

Projecting the given surface on the xoy plane we get $dx\,dy = \hat{n}\cdot\hat{k}\,ds$ *[Refer the definition of surface integral]*

$$\therefore \quad ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} = \frac{dx\,dy}{z/a} = \frac{a\,dx\,dy}{z} \qquad \ldots(2)$$

From (1) and (2) $\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \dfrac{3xyz}{a}\cdot\dfrac{a\,dx\,dy}{z} = \iint_R 3xy\,dx\,dy$

The region R of integration is the quadrant of the circle $x^2 + y^2 = a^2$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 3\int_{x=0}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx$$

$$= 3\int_{x=0}^{a} x\left[\frac{y^2}{2}\right]_{y=0}^{\sqrt{a^2-x^2}} dx = \frac{3}{2}\int_{x=0}^{a} x(a^2 - x^2)\,dx$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{3}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{3}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{3a^4}{8}$$

Thus $\displaystyle\iint_S (yz\,i + zx\,j + xy\,k)\cdot\hat{n}\,ds = \frac{3a^4}{8}$

---

24. *Using divergence theorem evaluate* $\int \vec{A} \cdot \hat{n}\, ds$ *where* $\vec{A} = x^3\, i + y^3\, j + z^3\, k$ *and* S *is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$

[*Note* : $\int \vec{A} \cdot \hat{n}\, ds$ *is to be understood as* $\iint_S \vec{A} \cdot \hat{n}\, ds$]

We have divergence theorem $\iint_S \vec{A} \cdot \hat{n}\, ds = \iiint_V \operatorname{div} \vec{A}\, d\, V$

$$\operatorname{div} \vec{A} = \nabla \cdot \vec{A} = 3x^2 + 3y^2 + 3z^2 = 3\,(x^2 + y^2 + z^2)$$

$$\therefore \quad \iint_S \vec{A} \cdot \hat{n}\, ds = \iiint_V 3\,(x^2 + y^2 + z^2)\, dx\, dy\, dz$$

Note : *The evaluation becomes highly tedious in the existing form with cartesion limits. Hence we evaluate by changing into spherical polar coordinates* (r, θ, ϕ)

Taking $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$ we have $x^2 + y^2 + z^2 = r^2$ and $dx\, dy\, dz = J\, dr\, d\theta\, d\phi$ where J is the jacobian of the transformation which works out to be $r^2 \sin\theta$. Further r varies from 0 to a, θ from 0 to π and ϕ from 0 to 2π

$$\therefore \quad \iint_S \vec{A} \cdot \hat{n}\, ds = 3 \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \cdot r^2 \sin\theta\; d\phi\; d\theta\; dr$$

$$= 3 \int_{r=0}^{a} \int_{\theta=0}^{\pi} r^4 \sin\theta\, [\,\phi\,]_0^{2\pi}\; d\theta\; dr$$

$$= 3 \int_{r=0}^{a} 2\pi\, r^4\, [\,-\cos\theta\,]_0^{\pi}\, dr$$

$$= -6\pi \int_{r=0}^{a} r^4\,(-2)\, dr = 12\pi \left[\frac{r^5}{5}\right]_0^a = \frac{12\,\pi\, a^5}{5}$$

Thus $\displaystyle \iint_S \vec{A} \cdot \hat{n}\, ds = \frac{12\,\pi\, a^5}{5}$

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25. *Evaluate* $\int_S (ax\,i + by\,j + cz\,k)\cdot \hat{n}\,ds$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = 1$

Let $\vec{F} = axi + byj + czk$

We have $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dV$

$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = (a + b + c)$

$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (a + b + c)\,dV = (a + b + c)\,V$

where V is the volume of the sphere with unit radius and $V = 4/3\cdot\pi r^3$ for a sphere of radius r.

Here since we have $r = 1$, $V = 4/3\cdot\pi$

Thus $\iint_S \vec{F}\cdot\hat{n}\,ds = \frac{4\pi}{3}(a + b + c)$

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26. *If* $\vec{F} = 2xy\,i + yz^2\,j + xz\,k$ *and S is the rectangular parallelopiped bounded by* $x = 0,\ y = 0,\ z = 0,\ x = 2,\ y = 1,\ z = 3$ *evaluate* $\iint_S \vec{F}\cdot\hat{n}\,ds$

>> By divergence theorem $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dV$

We have, $\text{div}\,\vec{F} = \nabla\cdot\vec{F} = (2y + z^2 + x)$

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3} (2y + z^2 + x)\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{1} \left[2yz + \frac{z^3}{3} + xz\right]_{z=0}^{3} dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{1} (6y + 9 + 3x)\,dy\,dx$$

$$= \int_{x=0}^{2} [3y^2 + 9y + 3xy]_{y=0}^{1}\,dx$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_{x=0}^{2}(12+3x)\,dx = \left[12x+\frac{3x^2}{2}\right]_{x=0}^{2} = 30$$

Thus $\displaystyle\iint_S \vec{F}\cdot\hat{n}\,ds = 30$

27. *Evaluate* $\int_S \int \vec{F}\cdot\vec{n}\,ds$ *given* $\vec{F} = x\,i + y\,j + z\,k$ *over the sphere* $x^2+y^2+z^2 = a^2$

>> We have Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dV$$

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}i+\frac{\partial}{\partial y}j+\frac{\partial}{\partial z}k\right)\cdot(x\,i+y\,j+z\,k)$$

ie., $\text{div}\,\vec{F} = 3$

$$\therefore \quad \iint_S \vec{F}\cdot\vec{n}\,ds = \iiint_V 3\,dV = 3V$$

where V is the volume of the sphere of radius 'a' represented by $x^2+y^2+z^2 = a^2$ is given by $4\pi a^3/3$

Thus $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,ds = 3\cdot\frac{4\pi a^3}{3} = 4\pi a^3$

Thus $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,ds = 4\pi a^3$

EXERCISES

1. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x^2-y^2)\,i + xy\,j$ where C is the arc of the curve $y = x^3$ from $(0, 0)$ to $(2, 8)$
2. If $\vec{F} = e^x \sin y\,i + e^x \cos y\,j$, find the circulation of $\vec{F}$ round the curve C where C is the rectangle with vertices $(0, 0)$ $(1, 0)$ $(1, \pi/2)$ $(0, \pi/2)$.
   What is your inference ?
3. Find the total work done in moving a particle in a force field $\vec{F} = 3xy\,i - 5z\,j + 10x\,k$ along the curve : $x = t^2+1,\ y = 2t^2,\ z = t^3$ from $t = 1$ to $t = 2$

4. If $\vec{F} = xy\,i - z\,j + x^2 k$, evaluate $\int_C \vec{F} \times d\vec{r}$ where C is the curve : $x = t^2,\ y = 2t,\ z = t^3$ from $t = 0$ to 1

5. Verify Green's theorem in a plane for $\int_C (y - \sin x)\,dx + \cos x\,dy$ where C is the triangle formed by the lines $y = 0,\ x = \pi/2$ and $y = 2x/\pi$

6. Using Green's theorem evaluate : $\int_C (\cos x \sin y - xy)\,dx + \sin x \cos y\,dy$ where C is the circle with centre origin and unit radius.

7. Verify Stoke's theorem for the vector function $\vec{F} = 2xy\,i + (x^2 - y^2)\,j$ over the circle $x^2 + y^2 = 1,\ z = 0$

8. Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\,i - 2xy\,j$ over the rectangle in the xoy plane bounded by $x = 0,\ x = 2a,\ y = 0\ \ y = 2b$.

9. Verify Gauss divergence theorem for $\vec{F} = 4xz\,i - y^2 j + yz\,k$ over the unit cube.

10. Use Gauss divergence theorem to evaluate $\iint_S \vec{F} \cdot \hat{n}\,ds$ where $\vec{F} = (x^2 - z^2)\,i + 2xy\,j + (y^2 + z^2)\,k$ where S is the surface of the cube bounded by $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0,\ z = 1$

ANSWERS

1. 824/21
2. Circulation $= 0$, $\vec{F}$ is irrotational
3. 303
4. $1/30 \cdot (-27i - 20j + 42k)$
5. $-(\pi/4 + 2/\pi)$
6. 0
7. 0
8. $-16ab^2$
9. 3/2
10. 3.